Pontifícia Universidade Católica
do Rio de Janeiro

Alex Avelino Carrasco Martinez

Demographics and Real Interest Rate in the US economy

Dissertação de Mestrado

Thesis presented to the Programa de Pós–graduação em Economia da PUC-Rio in partial fulfillment of the requirements for the degree of Mestre em Economia.

Advisor: Prof. Carlos Viana de Carvalho

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Prof. Carlos Viana de Carvalho
Advisor
Departamento de Economia – PUC-Rio

Prof. Andrea Ferrero
Department of Economics – Oxford University

Prof. Eduardo Zilberman
Departamento de Economia – PUC-Rio

Rio de Janeiro, November the 11th, 2019

Alex Avelino Carrasco Martinez

B.A. in Economics by Federico Villareal National University (Lima, Peru), 2014.

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Abstract

Carrasco Martinez, Alex Avelino; Carvalho, Carlos Viana de (Advisor). **Demographics and Real Interest Rate in the US economy**. Rio de Janeiro, 2019. 53p. Dissertação de Mestrado – Departamento de Economia, Pontifícia Universidade Católica do Rio de Janeiro.

I develop an overlapping generations model with life cycle wage profile (LCWP), age-dependent mortality rate, liquidity constraints, and nominal rigidities. The model is calibrated to capture US demographic transition, LCWP estimations, and other salient features of the US economy during 1950-2017. The model is then used to examine the relationship between demographics and real interest rates and the main transmission mechanisms in play. I find that the rapid increase in the working age population from 1950-1980s has significantly contributed to the rise of real interest rates. The reversion of this process together with the increase in life expectancy triggered a rapid decline in the interest rates ever since. The heterogeneity in the marginal propensity to consume among workers plays a major role in connecting these fertility and real interest rate movements.
In an additional exercise, due to the evidence on large life expectancy forecast errors, I introduce a learning process about longevity and find that it can significantly augment the relevance of demographic factors in explaining real interest rate movements. Finally, I find that the central banks' failure to recognize the relationship between demographics and interest rates can generate, due to unaccounted changes in the natural interest rate, inflation rate variations.

Keywords

Fertility rate; Life expectancy; Demographic transition; Real interest rate; Overlapping generations.

Resumo

Carrasco Martinez, Alex Avelino; Carvalho, Carlos Viana de. **Demografía e Taxa de Juros Real na economia dos EUA**. Rio de Janeiro, 2019. 53p. Dissertação de Mestrado – Departamento de Economia, Pontifícia Universidade Católica do Rio de Janeiro.

Eu desenvolvo um modelo de gerações sobrepostas com crescimento salarial ao longo do ciclo de vida (LCWP, por sua sigla em inglês), taxa de mortalidade dependente da idade, restrições de liquidez e rigidez nominal. O modelo é calibrado para capturar a transição demográfica dos EUA, estimativas de LCWP e outras características importantes da economia dos EUA durante o período 1950-2017. O modelo é usado para examinar a relação entre dados demográficos e taxas de juros reais assim como os principais mecanismos de transmissão em jogo. Eu encontro que o rápido aumento da população em idade ativa entre 1950 e 1980 contribuiu significativamente para o aumento das taxas de juros reais. A reversão desse processo, juntamente com o aumento da expectativa de vida, desencadeou um rápido declínio nas taxas de juros desde então. A heterogeneidade na propensão marginal a consumir entre os trabalhadores desempenha um papel importante na conexão desses movimentos de fertilidade e taxa de juros real. Num exercício adicional, devido à evidência de grandes erros de previsão da expectativa de vida, eu estendo o modelo com um processo de aprendizado sobre longevidade e encontro que ele pode aumentar significativamente a relevância de fatores demográficos na explicação dos movimentos reais das taxas de juros. Por fim, encontro que a falha dos bancos centrais em levar em conta a relação entre dados demográficos e taxas de juros pode gerar, devido a mudanças não monitoradas na taxa de juros natural, variações na taxa de inflação.

Palavras-chave

Taxa de fertilidade; Expectativa de vida; Transição Demográfica; Taxa de juros real; Gerações sobrepostas.

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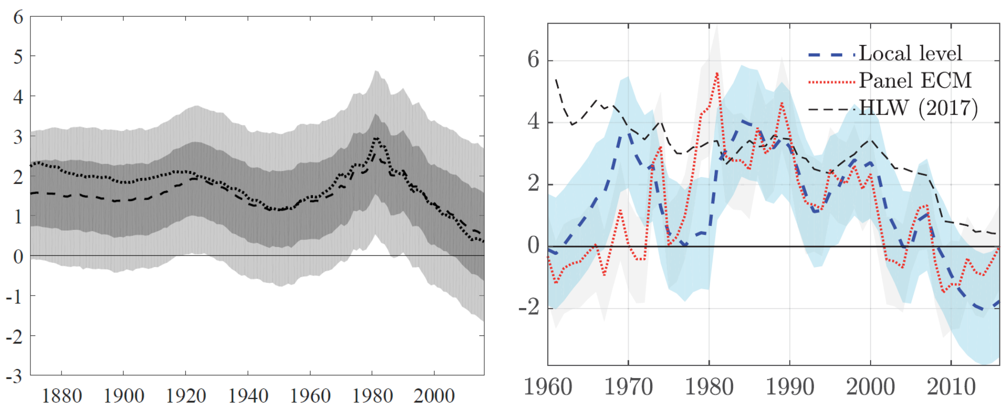
1
Introduction

To which extent do demographic trends help explaining US real interest rate variations? If they do, which are the main transmission mechanisms? Recent research has emphasized the role of demographics in the evolution of real interest rates.[1] However, to the best of my knowledge, there are only few works investigating the mechanisms through which demographic changes affect the real interest rate. In this paper, I contribute to close this gap in the literature. I develop an overlapping generations (OLG) model with life cycle wage profile (LCWP), age-dependent mortality rate, liquidity constraints, and nominal rigidities. The model is calibrated to capture US demographic trends, LCWP estimations, and other salient features of the US economy during 1950-2017. The model is finally used to theoretically address the above questions.

Figure 1.1: Documented Evolution of US Interest Rates



1.1(a): Trend in Real Interest Rate

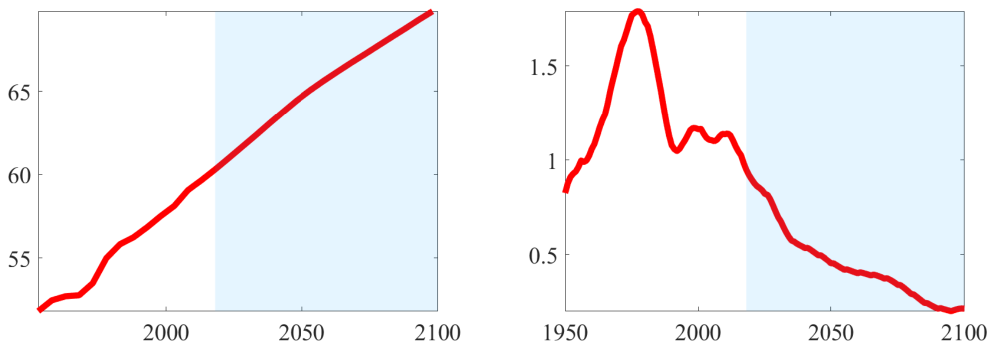
1.1(b): Natural Interest Rate

Note: Panel (a) The dashed black line shows the global real interest rate trend and the shaded areas show the 68 and 95 percent confidence interval. The dotted black line shows US real interest rate trend. Panel (b) Local level model estimates are in dashed blue with their 68% confidence interval in the shadowed light blue area, Panel ECM estimates are in dotted red jointly with 68% confidence interval in the shaded gray are, and estimates from Holston et al. (2017) in dashed black. **Source**: Del Negro et al. (2019) and Fiorentini et al. (2018).

[1]See for example Ferrero (2010), Backus et al. (2013), Carvalho and Ferrero (2015), Aksoy et al. (2015), Curtis et al. (2015), Favero et al. (2016), Carvalho et al. (2016), Curtis et al. (2017), Maurer (2017), Ferrero et al. (2017), and Sudo and Takizuka (2018).

Recent studies have documented the evolution of the real and natural interest rates for advanced economies.[2] Figure 1.1 summarizes the main findings of these studies. Those rates rose gradually after World War II, peaking in the 1980s, but they have been declining ever since. Clark and Kozicki (2005) and Lunsford and West (2017) found that, contrary to common knowledge, both productivity and trend growth seem to play a negligible role in driving these movements. In contrast, Del Negro et al. (2019) and Fiorentini et al. (2018) found that demographic transitions account for a large fraction of the movement of the real interest rate trend and the natural interest rate. As Figure 1.2 shows, there are two key features that characterizes demographics in USA: (i) rapid gains in life expectancy and (ii) a rise and fall in population growth. At first glance, one might suspect that population growth (or fertility rate) movements are the main responsible of the evolution of the real (and natural) interest rate since they seen to mirror each other. I find that this remark is true.

Figure 1.2: Demographic Transitions in USA



1.2(a): Life Expectancy at age 20 1.2(b): Population Growth

Note: Lines over shaded areas are forecasts. I measure population growth for people aged more than 20 years. **Source**: UN World Population Prospects 2017.

The model due to Gertler (1999), often used in the literature, emphasises three main channels through which demographic transitions can affect real interest rates. First, the LEX channel states that for a given retirement age, a rise in life expectancy (LEX) lengthens the retirement period and generates additional incentives to save, creating downward pressures on the real interest rate. Thus, the LEX channel predicts a lower real interest rate. Next, variations in the population growth may produce two opposite effects on real interest rate. On the one hand, the labor intensity channel indicates that an increment in

[2] See for example Hamilton et al. (2016), Del Negro et al. (2017), Fiorentini et al. (2018), and Del Negro et al. (2019).

the fertility rate will lead to a lower capital-labor unit ratio which will increase the marginal product of capital and the real interest rate. On the other hand, the population composition channel predicts that a higher fertility rate will drive down the dependency ratio and, due to heterogeneity in the marginal propensity to consume between workers and retirees, this composition change will push up aggregate saving and will decrease the real interest rate.

Nonetheless, those models ignore the presence of LCWP. The introduction of LCWP induces greater heterogeneity in the marginal propensity to save among workers: since young workers are less productive than middle-aged ones and they face liquidity constraints, the marginal propensity to consume is higher for the former. Thus, the model developed here helps us revisit both capital-labor unit and heterogeneity channels. Specifically, I find that the qualitative effects suggested by these channels change. First, the labor intensity channel indicates that, since young workers are less productive than middle-aged ones, an unexpected increase in the fertility rate reduces effective-labor force in per-capita terms, increments the capital-labor unit ratio, and lowers the real interest rate. Second, the population composition channel asserts that the rise in the fertility rate temporally increments the share of young workers and shrinks savings per-capita because the marginal propensity to consume is one of the highest in the economy. Thus, it makes stock of capital (in per-capita terms) lower and the real interest rate rises.

In this dissertation, I argue that including these features is crucial in order to obtain a significant relationship between demographics and real interest rates. In other words, the population composition channel plays a significant role to link the rise in population growth with the evolution of the real interest rate from 1950 to 1980s. Overall, I estimate that demographic factors can explain a rise and a reduction in the real interest rate around 1 and 3 percentage points respectively.

Furthermore, there is another salient feature in the US economy that can be relevant to address my initial questions above. Maurer (2017) and Lee and Tuljapurkar (1998) emphasise the challenge to predict life expectancy. They show evidence that official life-table projections were poor predictors for life expectancy during the period under analysis. I combine adaptive and eductive learning equilibrium concepts[3] in order to address this feature in an Heterogeneous Agent model. To the best of my knowledge, there are only few studies introducing learning in these kind of models.[4]

[3]See Evans and Honkapohja (2001), Evans and Honkapohja (2009), Eusepi and Preston (2011)

[4]See for example Farhi and Werning (2017), Qiu (2018), and Molavi (2019).

In the learning equilibrium I propose, agents learn about longevity movements adaptively, i.e., they should estimate the mortality generator process from the number of observed deaths. However, they form expectations about the future path of prices by solving a "perfect" foresight equilibrium conditional on their demographic estimations. Parameters which govern learning dynamics are calibrated to match observed life expectancy forecasts at different years during 1950-2017. Using this learning-about-longevity process, I find that the demographic contribution to the rise of the real and natural interest rate can be doubled since the evolution life expectancy is underestimated during the first years of analysis.

Last but not least, in line with Carvalho and Ferrero (2015), the presence of nominal rigidities lets us study the relation between demographics, natural interest rate, and inflation rate. The common view among central banks is that monitoring natural interest rate movements is fundamental in order to set optimal monetary policy. Specifically, interest rate deviations from its natural counterpart is the key measure of monetary policy stance and demand-side inflationary pressures (negative deviations can be interpreted as upward pressures for the output gap and inflation rate). Since demographic trends affect aggregate saving rates and the natural interest rate, the failure to account for them might trigger inflationary (or deflationary) episodes. Hence, I remark that central banks' miss-perception of initial rise of the natural interest rate in the 1950s can potentially explain an inflationary episode of 2 percentage points, and, as long as the natural interest rate started declining, a disinflationary episode around 2.5 percentage points.

The rest of this document proceeds as follows. Chapter 2 describes the OLG model. Chapter 3 explains the calibration strategy while Chapter 4 describes my main results in detail. Finally, Chapter 5 concludes.

2
A General Equilibrium Model

In this section, I describe the general equilibrium overlapping generations (OLG) model used to study the effect of demographic transitions on the real interest rate. The economy is closed, there is no aggregate economic uncertainty, time is continuous, and the lifetime is stochastic, i.e., people do not know when they are going to die. The are three types of agents in the economy: households, firms, and an infinitely lived government. Households can be divided in retirees and workers, firms in intermediate and final producers, and government conducts fiscal and monetary policy. For the rest of the exposition, small letters characterize individual variables while capital letters aggregated ones. I present a complete derivation of the model in Appendix A.

It is worth to emphasize that the main reason to use a continuous-time model is to permit rich heterogeneity along-life cycle (age-dependent mortality rate and wage profile) without significant increase in computational costs. I rely on Achdou et al. (2017) and Ahn et al. (2017)'s novel solution techniques for continuous-time models.

2.1
Households

The economy is populated by a continuum of individuals with total measure of N_t at time t. There are two types of individuals: N_t^r retirees and N_t^w workers. I assume that only workers can procreate and, at time t, the number of total births is $b_t N_t^w dt$ were b_t denotes the birth rate.

Preferences are time separable with subjective discount rate ρ and an instantaneous utility function given by $u(c) = \frac{c^{1-1/\eta}}{1-1/\eta}$ with $\eta > 0$. I also assume that an individual cares about total bequests left. Following De Nardi (2004) and De Nardi and Yang (2014), the utility from bequests $\mathcal{B}$ is denoted by $\underline{V}(\mathcal{B}) = \phi_1(\mathcal{B} + \phi_2 X_t)^{1-1/\eta}$ where X_t denotes the aggregate productivity.

Households savings can be invested in two different risk-free assets: capital k_t and government bonds b_t^g. In the absence of any market segmentation and aggregate uncertainty, by non-arbitrage condition, the return on both assets must be the same rate r_t. Hence, household's optimization problem can be written in terms of a one-dimensional state variable, $a_t(j) = k_t(j) + b_t^g(j)$.

In addition, I assume that workers can borrow assets up to an exogenous limit $\underline{a}X_t$ and that people born as worker with zero assets, i.e., $a_t(0) = 0$.

Retirees. At instant t, retirees aged j years choose a consumption plan $\{c^r_{t+s}(j)\}_{s\geq 0}$ and their preferences over time are

$$\mathbb{E}_t \left[\int_0^{T^d} e^{-\rho s} u\left(c^r_{t+s}(j)\right) ds + e^{-\rho T^d} \underline{V}\left(a^r_{t+T^d}(j)\right) \right] \tag{2-1}$$

where randomness comes from the unknown lifespan $j + T^d$ and $a^r_{t+T^d}(j)$ is the level of wealth accumulated until instant $t+T^d$. Mortality risk is modelled by the compensated Poisson jump process $J^d_t(j)$ with age-dependent intensity rate $\lambda^d_t(j)$.

Retiree's initial asset holdings upon retirement correspond to the asset held the previous instant as a worker. Moreover, they receive social security pension S_t and bequests ξ_t. Hence, retiree's financial wealth evolves according to

$$\dot{a}^r_t(j) = r_t a^r_t(j) + \xi_t + S_t - c^r_t \tag{2-2}$$

and subject to $a_t \geq -\underline{a}X_t$.

Retiree's problem can be formulated in a recursive way just as in discrete time problems. Let $V^r_t(a, j)$ be the value function for a retiree who aged j years and owns a assets, then its Hamilton-Jacobi-Bellman (HJB) equation is given by

$$\begin{aligned} 0 = \max_c \Big\{ & u(c) - \rho V^r_t(a, j) + \partial_a V^r_t(a, j) \left[r_t a + \xi_t + S_t - c_t \right] \\ & + \partial_j V^r_t(a, j) + \lambda^d_t(j) [\underline{V}(a) - V^r_t(a, j)] + \partial_t V^r_t(a, j) \Big\} \end{aligned} \tag{2-3}$$

subject to $a_t \geq -\underline{a}X_t$. Recursive formulation for worker's and retiree's problem is derived in Appendix A.1.

Workers. Similarly, at instant t, workers aged j years choose a consumption plan $\{c^w_{t+s}(j)\}_{s\geq 0}$ and their preferences over time are

$$\begin{aligned} \mathbb{E}_t \Big\{ & \mathbf{1}_{T^d < T^r} \left[\int_0^{T^d} e^{-\rho s} u\left(c^w_{t+s}(j)\right) ds + e^{-\rho T^d} \underline{V}\left(a^w_{t+T^d}\right) \right] \\ & + \mathbf{1}_{T^d \geq T^r} \left[\int_0^{T^r} e^{-\rho s} u\left(c^w_{t+s}(j)\right) ds + e^{-\rho T^r} V^r_{t+T^r}\left(a^w_{t+T^r}, j + T^r\right) \right] \Big\} \end{aligned} \tag{2-4}$$

where randomness is due to both the unknown lifespan $j + T^d$ and the retirement instant $j + T^r$. Hence, each worker not only faces mortality risk but also retirement uncertainty. Retirement risk is modelled by the compensated Poisson jump process $J_t^r(j)$ with age-dependent intensity rate $\lambda_t^r(j)$.

I introduce life cycle wage profile (LCWP) via age-dependent labour efficiency, $e(j)$. Thus, workers aged j years, earn total labor income $e(j)w_t$, receive bequests ξ_t, and pay lump-sum τ_t and social security ς_t taxes. Furthermore, firms distribute total profits Π_t to workers in proportion to productivity, i.e., a fraction $\tilde{e}_t(j) = \frac{e(j)}{\int e(j)N_t^w(j)dj}$, where $N_t^w(j)$ is the number of j years old workers. All in all, the law of motion for worker's financial wealth is

$$\dot{a}_t^w(j) = r_t a_t^w(j) + (1 - \varsigma_t)e(j)w_t + \tilde{e}_t(j)\Pi_t + \xi_t - \tau_t - c_t^w \tag{2-5}$$

For the sake of formulating the worker's recursive problem, let $V_t^w(a,j)$ be the value function for a worker who owns a assets and aged j years, then the worker's *HJB equation* is given by

$$\begin{aligned} 0 = \max_c \Big\{ & u(c) - \rho V_t^w(a,j) + \partial_a V_t^w(a,j)[r_t a + (1 - \varsigma_t)e(j)w_t + \tilde{e}_t(j)\Pi_t \\ & + \xi_t - \tau_t - c] + \partial_j V_t^w(a,j) + \lambda_t^r(j)[V_t^r(a,j) - V_t^w(a,j)] \\ & + \lambda_t^d(j)[\underline{V}(a) - V_t^w(a,j)] + \partial_t V_t^w(a,j) \Big\} \end{aligned} \tag{2-6}$$

subject to $a_t \geq -\underline{a}X_t$.

Consumption optimal rules for both workers and retirees, $c_t^r(a,j)$ and $c_t^w(a,j)$ respectively, imply drifts for financial assets and, together with stochastic processes $J_t^r(j)$ and $J_t^d(j)$, they induce joint measures of assets and age: $g_t^r(a,j)$ and $g_t^w(a,j)$.

2.2 Firms

The supply side of the model is standard in New-Keynesian framework. Two types of firms operate in the economy. A continuum of monopolistic competitive firms hire labor and rent capital from households to produce differentiated intermediate goods. Competitive retailers combine these intermediate goods to produce a homogeneous final good which is used for both consumption and investment.

Final-Goods Producers. A competitive representative final-good producer aggregates a continuum of intermediate inputs indexed by $i \in [0,1]$

$$Y_t = \left(\int_0^1 y_{t,i}^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{2-7}$$

where $\varepsilon > 0$ is the elasticity of substitution across goods. Cost minimization yields the demand for i^{th} intermediate good as function of its relative price and total demand

$$y_{t,i} = \left(\frac{p_{t,i}}{P_t}\right)^{-\varepsilon} Y_t \tag{2-8}$$

where $P_t = \left(\int_0^1 p_{t,i}^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}$ is the price for one unit of final good.

Intermediate Goods Producers. Intermediate goods are produced using capital, $k_{t,i}$, and effective-labor units, $l_{t,i}$, according to a standard Cobb-Douglas labor-augmenting technology

$$y_{t,i} = k_{t,i}^{1-\alpha}[X_t l_{t,i}]^{\alpha} \tag{2-9}$$

where $\alpha \in (0,1)$ is the labor share and the technology factor X_t grows deterministically at rate μ_t^x

$$dX_t = \mu_t^x X_t dt$$

Let $N_{t,i}^w(j)$ be the number of workers aged j years hired by firm i, then the effective-labor unit, $l_{t,i}$, is the aggregated level of labor productivity: $\int e(j)N_{t,i}^w(j)dj$. Otherwise, cost minimization implies that the marginal cost is common across all producers and given by

$$m_t = \left(\frac{r_t + \delta_k}{1-\alpha}\right)^{1-\alpha}\left(\frac{w_t/X_t}{\alpha}\right)^{\alpha} \tag{2-10}$$

where factor prices is equal their respective marginal revenue products. Each intermediate producer i has monopolistic power and maximizes profits subject to price adjustment costs as in Rotemberg (1982). Hence, at t they choose $\{p_s\}_{s\geq t}$ to maximize

$$\int_t^{\infty} e^{-\int_t^s r_\tau d\tau}\left\{\tilde{\Pi}_s(p_s) - \Theta_s\left(\frac{\dot{p}_s}{p_s}\right)\right\}d\tau$$

where

$$\tilde{\Pi}_s(p_s) = \left(\frac{p_s}{P_s} - m_s\right)\left(\frac{p_s}{P_s}\right)^{-\varepsilon} Y_s \qquad \text{and} \qquad \Theta_s(x) = \frac{\theta}{2}x^2 Y_s$$

The solution for this pricing problem yields the exact New Keynesian Phillips

Curve which characterized the evolution of inflation rate $\pi_t = \frac{\dot{P}_t}{P_t}$

$$\dot{\pi}_t = \left[r_t - \frac{\dot{Y}_t}{Y_t} \right] \pi_t - \frac{\varepsilon}{\theta}(m_t - m^\star) \tag{2-11}$$

where $m^\star = \frac{\varepsilon - 1}{\varepsilon}$ is the inverse of the flexible price optimum mark-up level.

2.3 Government

Fiscal Authority. The government issues short-term debt B_t^g and levies lump-sum taxes to finance a given stream of spending $\{G_t\}_{t\geq 0}$. Thence, the total stock of government's bond evolves according to

$$\dot{B}_t^g = r_t B_t^g + G_t - N_t^w \tau_t \tag{2-12}$$

Social security is based on pay-as-you-go system. Then the aggregate social security tax revenue is equally distributed among retirees

$$N_t^r S_t = \varsigma_t w_t \int e(j) N_t^w(j) dj \tag{2-13}$$

At any given moments, bequests are fully taxed by the government and then redistributed uniformly to all living agents. The total amount of bequests is given by

$$\xi_t N_t = \int_0^\infty \lambda_t^d(j) \left\{ \sum_{i \in \{w,r\}} \int a g_t^i(a,j) da \right\} dj \tag{2-14}$$

Central Bank. The monetary authority sets the nominal interest rate in agreement with a standard Taylor rule,

$$\exp(i_t) = \exp(r_t^i) \exp(\phi_\pi \pi_t) \tag{2-15}$$

where r_t^i is a Taylor Rule's drift and $\phi_\pi > 1$.

2.4 Competitive Equilibrium

A competitive equilibrium in this economy is defined as paths for individual household and firm decisions $\{c_t^r, c_t^w, a_t^r, a_t^w, l_t, k_t\}_{t\geq 0}$, input prices $\{w_t, r_t\}_{t\geq 0}$, inflation rate $\{\pi_t\}_{t\geq 0}$, fiscal variables $\{\tau_t, \varsigma_t, G_t, B_t^g\}_{\geq 0}$, measures $\{g_t^w, g_t^r\}_{t\geq 0}$, and aggregate quantities such that, given the exogenous demographic process $\{b_t, \lambda_t^r, \lambda_t^d\}_{t\geq 0}$, at every t: (i) households and firms maximize their objective functions taking as given equilibrium prices, taxes, and transfers; (ii) measures are consistent with Kolmogorov Forward Equation[1]; (iii) the government budget constraints holds; and (iv) all markets clear. There are three markets in the economy: asset, labor, and goods market.

The asset market clear when physical capital K_t plus the aggregate government bonds B_t^g equals household's holdings of assets $A_t \equiv \int\int a g^w(a,j)dadj + \int\int a g^r(a,j)dadj$, i.e.,

$$K_t + B_t^g = A_t \tag{2-16}$$

The labor market clears when the effective labor hired by intermediate good producers equals the aggregate supply of effective labor

$$L_t \equiv \int_0^1 l_{t,i}di = \int_0^\infty e(j)N_t^w(j)dj \tag{2-17}$$

Finally, goods market clearing conditions is that

$$Y_t = C_t + I_t + G_t + \Theta_t \tag{2-18}$$

where $C_t \equiv \int\int c^w(a,j)g^w(a,j)dadj + \int\int c^r(a,j)g^r(a,j)dadj$ and $I_t = \dot{K}_t + \delta_k K_t$.

[1]The Kolmogorov Forward Equation states the evolution of joint measures g_t^w and g_t^r based aggregate consistency conditions. For details, see Appendix A.2.

3
Taking the Model to Data

I have three broad goals in choosing the parameters of the model. First, I need to define hazard ratios $\lambda_t^i(j)$ for $i \in \{r, d\}$ and fertility rate b_t in order to match US demographic transition. Second, I seek for a calibration of labor productivity $e(j)$ which is consistent with empirical life cycle wage profile estimation for the US economy. Finally, I calibrated some parameters for the sake of matching salient features of the US economy and use well accepted calibration in New Keynesian literature for the rest of the parameters.

Demographic Transitions. Human Mortality Database (HMD) contains estimated US life-tables from 1950 to 2017 in annual frequency. Projections from 2018 to 2100 are collected in the United Nations World Population Prospects 2017 (UN) in 5-year frequency. Then, for the sake of gathering as much observations as possible, I combine HMD and UN and obtain empirical estimations for US life-tables during 1950-2100.

Since I am not empirically interested in the effects of changes in the retirement age, and for simplicity, I set

$$\lambda_t^r(j) = \begin{cases} 0, & j < 45 \\ \mathcal{R}, & j \geq 45 \end{cases} \tag{3-1}$$

where $\mathcal{R}$ is a large positive number. Contrarily, I use a two stage procedure to calibrate mortality process. The first stage consists in estimating the parsimonious Lee and Carter (1992) model using empirical mortality rate $\mathcal{M}_t(j)$,

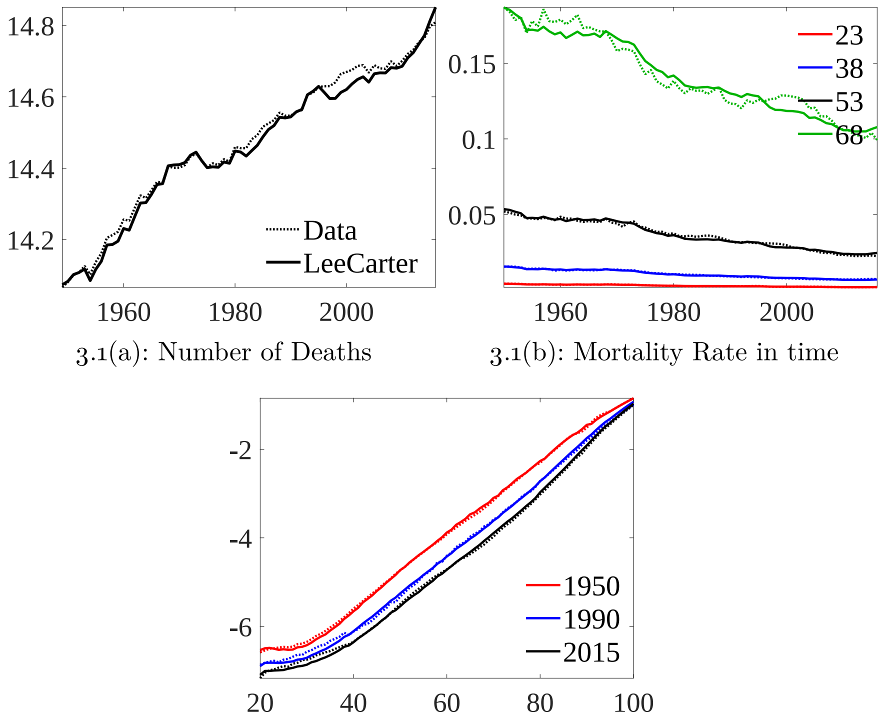
$$\ln \mathcal{M}_t(j) = \upsilon_0^d(j) + \upsilon_1^d(j)\mathcal{K}_t^{d,\text{LC}} + u_t^{\text{resid}} \tag{3-2}$$

In this equation, $\upsilon_1^d(j)$ tell us which rates decline rapidly or slowly in response to changes in the Lee-Carter mortality index, $\mathcal{K}_t^{d,\text{LC}}$. Figure 3.1 suggests that estimations match quite well US number of deaths to mortality rate.

Secondly, I approximate the estimated Lee-Carter mortality index using two distinct Ornstein-Uhlenbeck (OU) processes:

$$d\mathcal{K}_t^d = \mathcal{E}_{1,t}^d dt - \mathcal{E}_{2,t}^d dt \tag{3-3}$$

Figure 3.1: Mortality Rate Estimation (log-scale)



3.1(a): Number of Deaths

3.1(b): Mortality Rate in time

3.1(c): Mortality Rate in age

Source: Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org or www.humanmortality.de (data downloaded on December, 2018).

$$d\mathcal{E}^d_{i,t} = -\psi^i \mathcal{E}^d_{i,t} dt + u^i_t \tag{3-4}$$

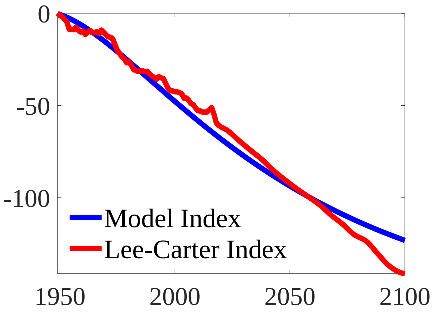
for $i \in \{1, 2\}$. Thus, I calibrate $(\psi^1, \psi^2, u^1_0, u^2_0)$ to minimize distances between $\mathcal{K}^{d,\mathrm{LC}}_t$ and $\mathcal{K}^d_t$. Finally, the mortality rate used in the model, $\lambda^d_t(j)$, is computed using estimations for $\upsilon^d_0(j)$ and $\upsilon^d_1(j)$ (First Stage) and $\mathcal{K}^d_t$ (Second stage)

$$\ln \lambda^d_t(j) = \upsilon^d_0(j) + \upsilon^d_1(j)\mathcal{K}^d_t \tag{3-5}$$

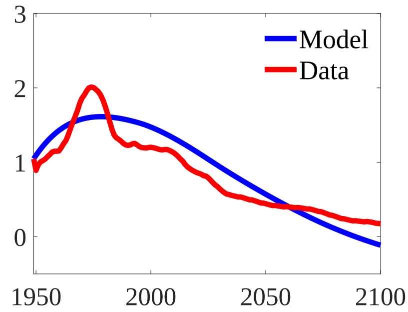
Fertility process b_t is the sum of a constant υ^b and a time-varying process $\mathcal{K}^b_t$ which is similar to the modelled mortality index, $\mathcal{K}^d_t$. Hence, parameters of $\mathcal{K}^{b,t}$ are calibrated to match historical population growth. Figure 3.2 shows the demographic calibration implies demographic transitions in the model together with historical (and projected) demographic trends.

Furthermore, I assume that long-run mortality and fertility rates are different to their initial values, i.e., a non-stationary demographic transition. It is worth emphasizing that the model is able to reproduce empirical age distributions and mortality rate for both the initial (1950) and final (2100) steady-state (see Figure 3.3).

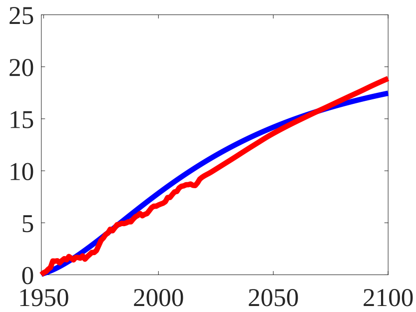
Figure 3.2: Demographic Transitions



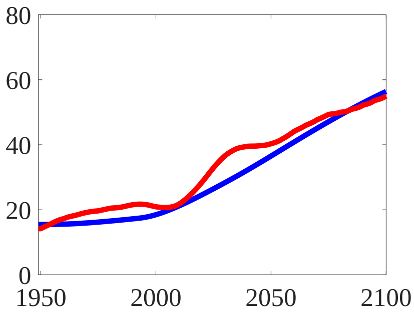
3.2(a): Mortality Index



3.2(b): Population Growth



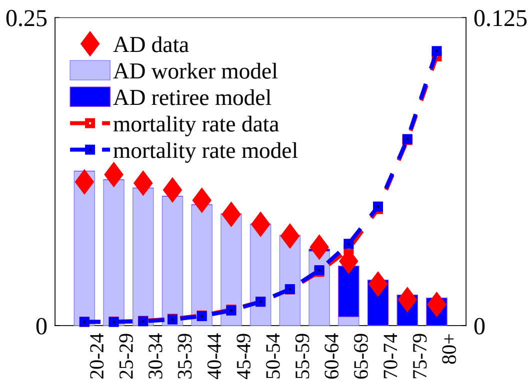
3.2(c): Life Expectancy (Dev.)



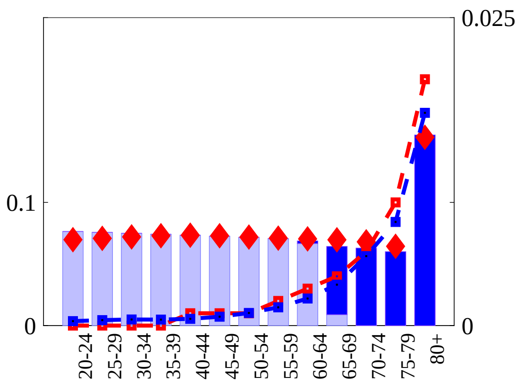
3.2(d): Dependency Ratio

Note. Projections start at 2018. Life expectancy is expressed in deviations from its initial value. **Source**: Human Mortality Database and UN World Population Prospects 2017.

Figure 3.3: Age Distribution and Mortality Rates



3.3(a): Demographics, year 1950



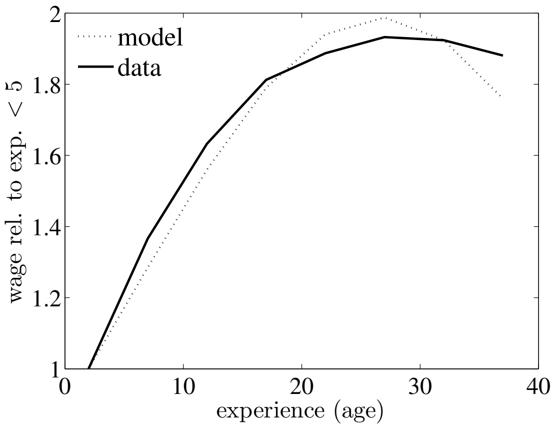
3.3(b): Demographics, year 2100

Labor Productivity (Life Cycle Wage Profile - LCWP). For the age-dependent labor productivity $e(j)$, I specify a log-quadratic function

$$e(j) = \exp(e_a j^2 + e_b j + e_c) \tag{3-6}$$

Then, I choose $\{e_a, e_b, e_c\}$ such that $e(j)$ roughly match the US lyfe-cycle wage profile presented in Lagakos et al. (2018). Figure 3.4 shows the calibrated and empirical life cycle wage profiles.

Figure 3.4: Life Cycle Wage Profile



Note. I use the the ratio of average wages for workers in each 5-experience bin relative to the average wages for workers with less than 5 years of experience.

Preferences and borrowing limit. I set the elasticity of intertemporal substitution η to 0.25 which is consistent with estimates of Hall (1988) and Yogo (2004). The subjective discount rate ρ calibration is based on attaining a 4% p.a. in real interest rate. I closely follow De Nardi and Yang (2014)'s calibration for parameters in the utility from bequests $\underline{V}(\mathcal{B})$, i.e., $\phi_1 = -100$ and $\phi_2 = 10$. Borrowing limit $\underline{a}$ is set at zero, $\underline{a} = 0$, in line with evidence on liquidity constraints, see for example Hubbard and Judd (1986), Jappelli (1990), and Jappelli and Pistaferri (2010).

Production. Labor share of output equals 0.67 in line with national accounts. As in Kaplan et al. (2018), I set the elasticity of substitution for final goods to 10, implying a steady-state mark-up of 11%, and the constant θ in the price adjustment cost function to 100. Finally, capital depreciation rate is calibrated at 10% annually.

Government and Social Security. Government consumption and debt rates are set to their historical means during $1950 - 2017$, 15% and 40% respectively[1]. The social security tax ς is calibrated such that the average replacement rate is equal to 40% as it is common in social security studies (e.g., see De Nardi (2004)). Taylor rule coefficient ϕ_π equals 1.50 as commonly calibrated in New Keynesian models.

Along the following section I compare different calibrated versions in order to examine the role of some relevant features in the model. Table 3.1 summarizes the main characteristics of these distinct versions. The third col-

[1] I use government debt held by the public and not the total debt because the former is consistent to the concept of debt in the model.

umn express these characteristics in terms of parameter calibration. Note that ϕ_π equals a large positive number, i.e. $\phi_\pi \to \infty$, in order to generate equilibriums characterized by full inflation stabilization. All parameters ignored in this column are assumed to be equal to the baseline calibration. Finally, the baseline monetary calibration is summarized in Table 3.2.

Table 3.1: Models used in the Analysis

Name	Description	Calibration
Real	Baseline Frictionless Model	$\phi_\pi \to \infty$, $\epsilon \to \infty$
Non-LCWP	Frictionless Model without LCWP	$\phi_\pi \to \infty$, $\epsilon \to \infty$, $e(j) = 1$
Natural	Baseline Natural Model/Monopolistic Competition	$\phi_\pi \to \infty$
Nominal	Baseline Monetary Model	-

Table 3.2: Baseline Calibration for Monetary Model

	Description	Value	Target/Source
	Demographics		
b	fertility rate	see text	US Demography
$\lambda^i(j)$	i's hazard ratio	see text	US Demography
$e(j)$	Labor productivity	see text	US LCWP
	Preferences		
η	Elast. of Intertemporal Substitution	1/4	Yogo (2004)
ρ	Subjective discount rate[a]	-	Interest rate 4.0% (p.a.)
ϕ_1	$\underline{V}$'s parameter 1	-100	De Nardi and Yang (2014)
ϕ_2	$\underline{V}$'s parameter 2	10	De Nardi and Yang (2014)
	Unsecured borrowing		
$\underline{a}$	Borrowing limit	0	Liquidity Constraint
	Production		
α	Labor share	0.67	Carvalho et al. (2016)
δ	Depreciation Rate (p.a.)	10%	
ε	Deman elasticity	10	Kaplan et al. (2018)
θ	Price adjustment cost	100	Kaplan et al. (2018)
μ^x	Ss productivity growth	0.02	2.0% Per-Capita Growth
	Government		
b_y	Debt (% GDP)	40%	Av. during 1950 – 2017
g_y	Gov. expenditure (% GDP)	15%	Av. during 1950 – 2017
ς	Social Security Tax[b]	-	Av. replacement rate 40%
ϕ_π	Taylor rule coefficient	1.50	NK Literature

[a] Internally calibrated.
[b] Internally calibrated.

4
Quantitative Results

In this section, I examine the role of demographic trends in explaining real interest rate movements. The main result is that, due to heterogeneity in the marginal propensity to consume among workers, the observed hump-shaped in the evolution of fertility rate can explain the rise and fall in real and natural interest rate trends documented by Fiorentini et al. (2018) and Del Negro et al. (2019).

I divide this section in four parts. I start inspecting impulse-responses to a fertility rate shock and gaining some insights on its transmission channels. Next, I revisit transmission mechanisms of demographics and restate the relevance of the heterogeneity in marginal propensity to consume among workers. Then, adopting a learning equilibrium concept, I inspect the role of a learning process about longevity and decompose the contribution of each demographic factor on real interest rate movements using non-monetary models. Finally, a monetary model is used to shed light on the potential relevance of demographics in causing inflation rate movements.

4.1
Fertility Rate Shocks: A First Step in the Analysis

Before presenting the results of the main experiment, I inspect impulse-responses to fertility rate shocks in a stationary model. The analysis is useful to shed first lights and gain insights about transmission mechanisms through which demographics operate.

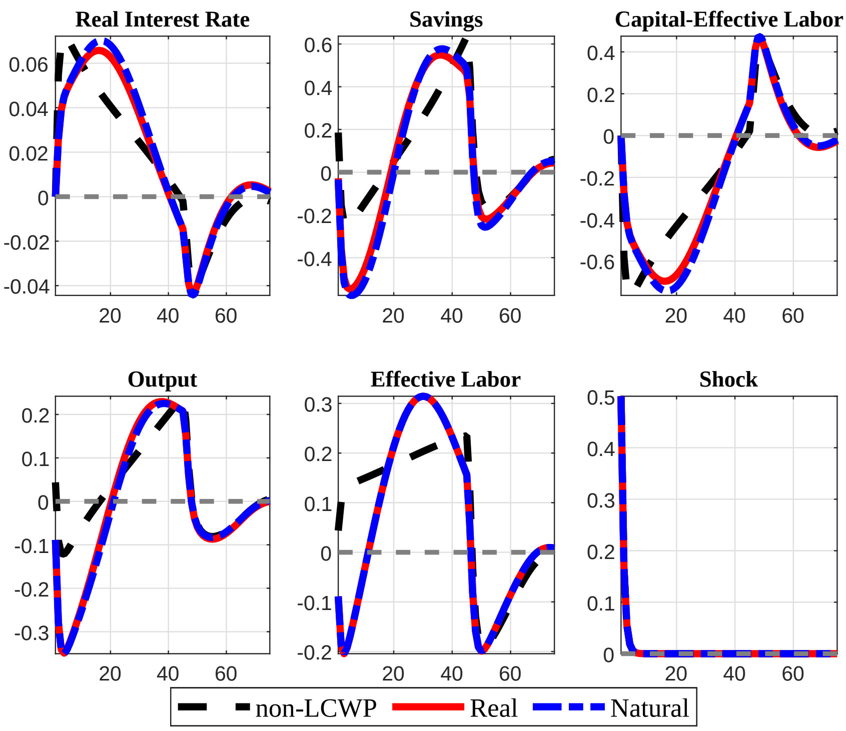
To begin with, I examine the consequences of introducing life cycle wage profile (LCWP) in the model. Figure 4.1 plots responses to an unanticipated 0.5 percentage point increase in the fertility rate across distinct calibrations. Responses of the effective-labor unit **in per-capita terms** $\frac{L_t}{N_t}$ are notoriously different across models. In a model without LCWP, the per-capita effective-labor unit, which equals $\frac{N_t^w}{N_t}$ in this scenario, increases because the rise in fertility rate directly augments the number of workers in the economy. However, in LCWP models, the same labor measure declines since productivity of young workers is lower than middle-aged workers. Surprisingly, positive fertility rate shocks induce an increment in the real interest rate for both type of models.

This suggests that there are different mechanisms in play.

In non-LCWP model, the fertility rate shock operates through an increase in the per-capita effective-labor unit which is consistent with a contraction in the capital-effective labor ratio and an increment in the real interest rate. In contrast, the heterogeneity in the marginal propensity to consume (MPC) plays an important role for LCWP models. The expansion in the mass of young workers, with higher MPC, reduces aggregate savings and the aggregate stock of capital, and the real interest rate rises. In fact, the response of aggregate savings is quite similar to responses of the effective-labor unit in these models (Real and Natural). Moreover, in Natural model, middle-aged workers receive a greater fraction of firm's profits and this increments their contribution to aggregate savings. Since the share of middle-aged workers falls, the real interest rate rises more than in Real model after a positive fertility rate shock.

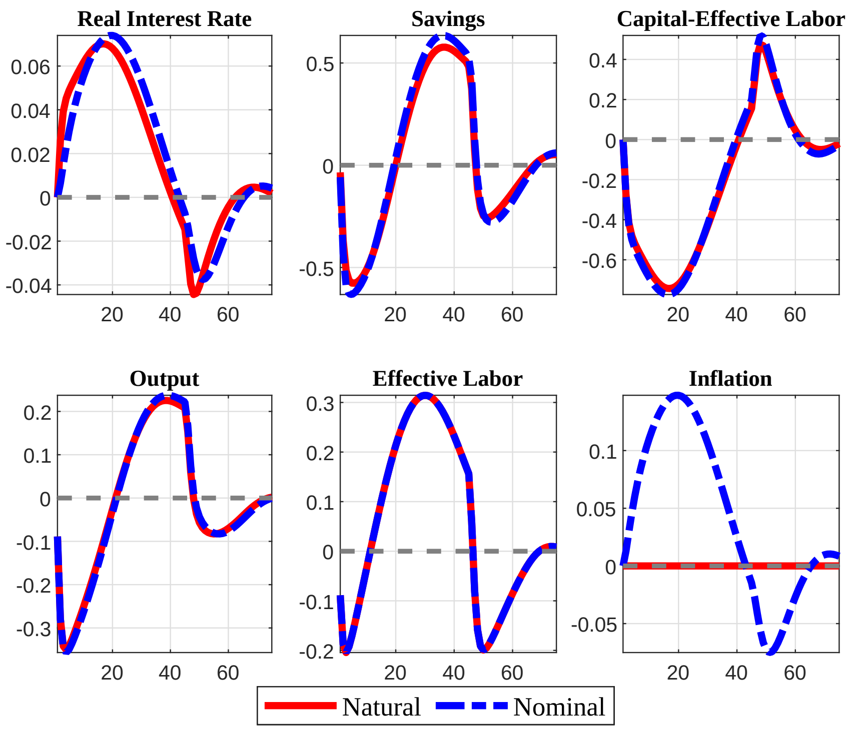
It is worth noting that the effects of a temporal increase in fertility rate are long-lasting. After 45 years, the previous commented effects reverse as long as the *baby boom generation* retires. The retirement of this huge mass of workers contracts aggregate savings because the marginal propensity to consume is higher for retirees than for workers. Nonetheless, real interest rate shrinks since labor force suffers a large contraction and the capital stock per-labor unit increases.

Figure 4.1: Fertility Rate Shock and LCWP



Whenever the central bank does not implement Taylor Rule drift r_t^i equals to the natural interest rate (real interest rate response in Natural model), fertility rate shocks generate inflation rate movements. This remark motivates the second impulse-responses analysis. Figure 4.2 illustrates the nominal effects for a myopic monetary policy: $r_t^i = r_{\text{steady-state}}$. In this context, fertility rate shock is inflationary during the first 45 years. As the natural interest rate increases and the central bank does not respond to these movements, the monetary policy stance becomes expansionary causing an inflationary period. As I have already pointed, the situation reverse when the *baby boom generation* retirees. This mechanism connects demographic trend and inflation rate variations and is in line with Carvalho and Ferrero (2015).

Figure 4.2: Fertility Rate Shock and Inflation

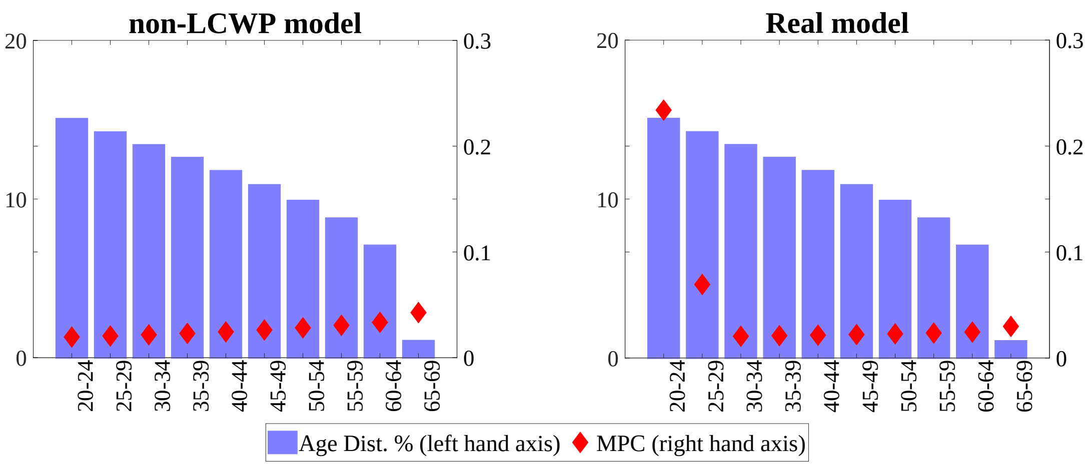


4.2
Non-Monetary Models: US Demographics and Real Interest Rate

Carvalho et al. (2016) emphasises three main channels through which demographic transitions affect real interest rates. First, they found the LEX Channel: for a given retirement age, a rise in life expectancy (LEX) lengthens the retirement period and generates additional incentives to save, creating downward pressure on the real interest rate. Next, the variations in the population growth may produce two opposite effects on real interest rate. On the one hand, the labor intensity channel states that an increment in fertility rate will lead to a lower capital-effective labor ratio which will increase marginal product of capital and the real interest rate. On the other hand, the population composition channel predicts that higher fertility rate will drive down the dependency ratio and, due to heterogeneity in MPC between workers and retirees, this composition change pushes up aggregate saving and will reduce the real interest rate.

Nonetheless, MPC's heterogeneity considered in Carvalho et al. (2016) and Gertler (1999) comes from differences between workers and retirees but not within each group. The model developed above is rich enough to capture heterogeneity in each of these groups. Figure 4.3 compares the worker's MPC heterogeneity[1] for both non-LCWP and Real[2] model at 1950. First, note that age-dependent mortality rate can generate heterogeneity in MPC by its own because it directly changes the effective subjective discount rate. However, there are not large differences since mortality rate are relatively low for the first 45 years in life (see Figure 3.3). Contrarily, the introduction of LCWP leads to significant heterogeneity: since young workers are less productive than middle-age workers, and they face liquidity constraints, the marginal propensity to consume is higher for the former. Furthermore, this MPC profile is consistent with empirical evidence on consumption responses (see for example Jappelli (1990) and Jappelli and Pistaferri (2010)) and leads to substantial change in the transmission mechanisms of demography as it has been already suggested in Section 4.1.

Figure 4.3: Heterogeneity in Worker's MPC at 1950



[1]Using the normalized model presented in Appendix A, the life-cycle consumption profile for $i \in \{w, r\}$ is computed as

$$\overline{C}_t^i(j) = \int_{-\underline{a}}^{\infty} \hat{C}_t^i(\hat{a}, j)\hat{g}_t^i(\hat{a}, j)d\hat{a}$$

Then, the MPC along the life cycle can be computed as

$$\mathsf{MPC}_t^i(j) \equiv \int_{-\underline{a}}^{\infty} \frac{d\overline{C}_t^i(\hat{a}, j)}{d\hat{a}}\hat{g}_t^i(\hat{a}, j)d\hat{a}$$

[2]MPC's distribution for Natural and Nominal model are quite similar.

Reinspecting the Mechanisms. The impulse-response analysis is helpful to gain some insights in the different mechanisms operating after demographic shocks. I will use these insights in order to analyse the effects of simulated demographic transitions and revisit some transmission mechanisms.

Figure 4.4 compares the simulated path for macroeconomic variables driven by demographic transitions for non-LCWP and Real model. On the one hand, non-LCWP model suggests that US demographic trends have produced a steadily decline in the real interest rate since 1950. This result implies that fertility rate channels play a negligible role in the connection of demographics and interest rates for this model. On the other hand, a model that generates more realistic MPC's heterogeneity can induce the documented rise and fall movement in the real interest rate as a consequence of US demographic transitions.

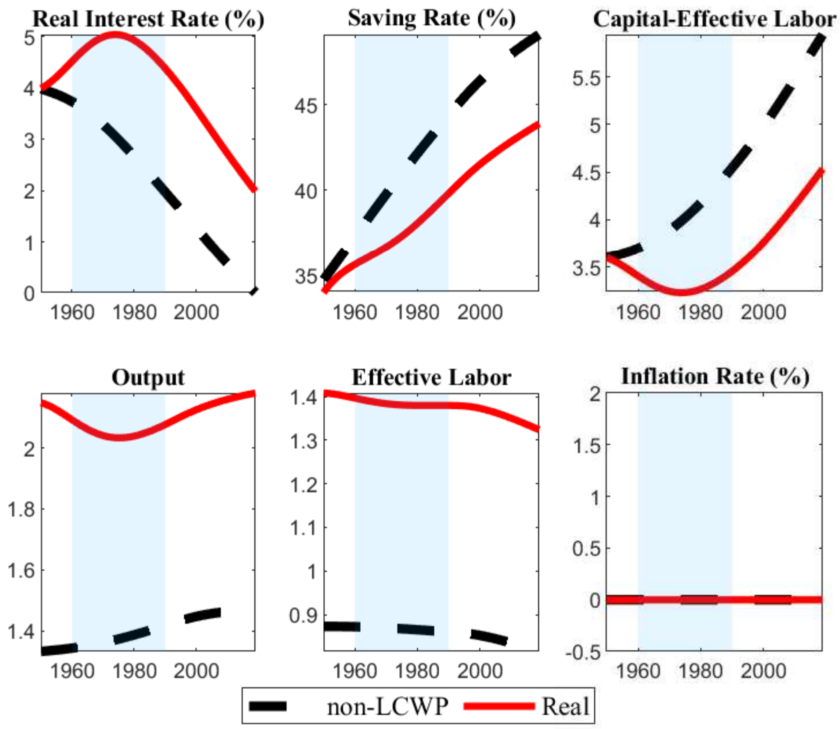
All in all, I revisit the transmission mechanisms for fertility rate shocks. First, the labor intensity channel (revisited) states that an unexpected increase in fertility rate contracts the effective-labor force in per-capita terms, then the capital-effective labor ratio increases and real interest rate falls. And secondly, the population composition channel (revisited) predicts that the initial increment in the share of young workers in the economy reduces aggregate savings and the subsequent reduction in the stock of capital pushes up the real interest rate. The hump-shaped trajectory in the real interest rate suggests that, for LCWP models, the population composition channel is quantitative more important than the labor intensity channel.

4.3
Adaptive-Eductive Learning Equilibrium (AELE): Learning about longevity

Up to this section, I have assumed that the agents know all the future path for demographics and equilibrium prices, i.e. I used perfect foresight equilibrium (PFE) concept. However, Maurer (2017) and Lee and Tuljapurkar (1998) emphasise the challenge to predict the life expectancy. They show evidence that official life-table projections were poor predictors of the observed life expectancy from 1950. Here, I use a different equilibrium concept and show that this learning-about-longevity process almost doubles the effect of US demographics in the initial rise of the natural interest rate.

Recall that mortality rate variations in the model are determined by parameters $(\psi^1, \psi^2, u_0^1, u_0^2)$. I assume that agents can observe the initial shocks in OU process (u_0^1 and u_0^2)), however they must estimate (ψ^1, ψ^2) using observed data. The agents update their estimations based on a constant Kalman gain rule Γ (**adaptive or econometric learning**),

Figure 4.4: Demographics and Macroeconomic Equilibrium



Note. Shaded areas show 1960-1990 period.

$$d\hat{\psi}_t^i = -\Gamma[\hat{\psi}_t^i - \psi_t^i] \tag{4-1}$$

for $i \in \{1, 2\}$. At moment t, people forecast the entire evolution of mortality rates $\{\lambda_s^{d,e}(j)\}_{s>t}$ using the estimated demographic generator process. Expectations about the future path of prices $\{r_s^e, w_s^e\}_{s>t}$, and the rest of the variables, are formed **eductively**, i.e. they equal the trajectory obtained in a PFE which assumes that mortality rates evolve as expected. Thus, whenever agents update parameter's estimation and their mortality rate forecast, they must "mentally" compute a kind of PFE which I call a **Point-wise Constrained Foresight Equilibrium (PCFE)**. Hence, the Adaptive-Eductive Learning Equilibrium is based on a continuous sequence of PCFE[3].

I calibrate Γ, ψ_0^1, and ψ_0^2 in order to match official life expectancy forecasts at three different years: at 1965 (based on Bayo (1966)), 1988 (based on Wade (1989)), and 2017 (based on UN-WPP 2017).

Demographic Contributions: Demographics and Natural Interest Rate. Natural interest rate movements can be measure in a model with monopolistic competition and no nominal rigidities, i.e., Natural model. Real

[3]Further details and formalization of this equilibrium concept, see Appendix D.

interest rate deviations from the initial steady-state and demographic factors contribution for both PFE and AELE solution using Natural model are plotted in Figure 4.5. Each demographic contribution is computed by subtracting simulation results based on distinct demographic transition: only fertility rate, all demographic transitions (fertility and LEX), and all demographics with AELE. For instance, the contribution of LEX movements in the evolution of natural interest rate is obtained subtracting r_T in all demographics simulation from r_t in only fertility rate simulation (both of them using PFE).

For PFE solution, the effect of demographics on real interest rates (black dotted line) steadily rose from 1950 to late-1970s reaching one percent increment at the end of this period. The behaviour of the interest rate during this period was mainly driven by fertility rate movements (blue bar), i.e., by a huge increase in the share of young workers which reduced aggregate savings and induced a rise in real interest rate. From the late-1970s to the present, the real interest rate has dropped 3 percentage points (p.p.) in total. Both demographic factors have contributed to this result. The positive contribution of fertility decreased as long as the population growth converge toward to its new steady-state value. Likewise, the contribution of the increase in life expectancy (white bars) has been consistently and increasingly negative during all the period under analysis.

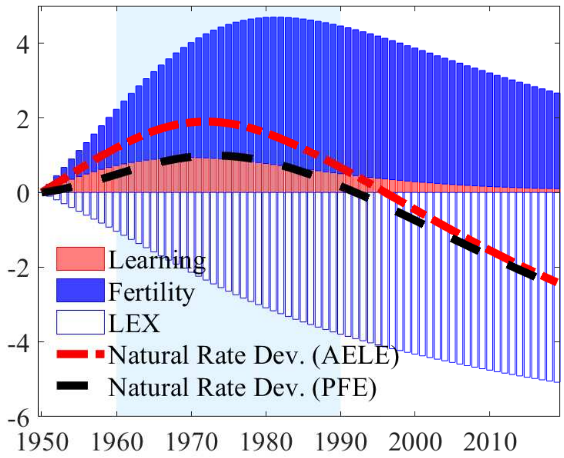
Figure 4.5 also shows the role of the learning-about-longevity process (light red bars). The partial ignorance about DGP's mortality rate conduces people to underestimate the initial increment in the life expectancy and it permits a higher increase in the natural interest rate (dotted red line) relative to PFE models. Specifically, this learning channel can double the impact on the natural interest rate at mid-1970s. From this peak, the real interest rates reduction exceeds 4 p.p. when I consider the learning process and around 3 p.p. from 1990.

4.4 Monetary Model: Real Interest Rate and Inflation

Using a monetary model I explore to which extent US demographic factors have contributed to the observed rise and fall in the inflation rate during 1950-2017. I have already shown that the dynamics in fertility rate and life expectancy contributed with the rise and fall of the real interest rate in the period under analysis, however, there is a question I have not answered yet: how has the real interest dynamics affected monetary policy and inflation?

The answer to this questions relies on the way the central bank decide about r_t^i, i.e., the Taylor Rule Drift. As long as the central bank follows the

Figure 4.5: Demographic Factors and Natural Interest Rate



Note. The dotted black line shows the variation in the real interest rate in a PFE's world, blue bars represent the contribution of fertility rate on interest rate movements, and white bars display the contribution of life expectancy variations. The dotted red line shows evolution of the real interest rate (deviations from the initial steady-state) in a AELE's model and light red bars display the contribution of learning-about-longevity process on those variations. Shaded areas show 1960-1990 period.

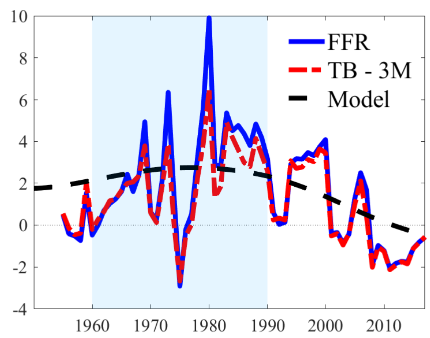
variations implied for Natural model, i.e. $r_t^i = r_t^{\text{Natural}}$, then the inflation rate would remain at zero for every moment. However, whenever the monetary authority does not internalize, or underestimate, the role of demographics on the natural interest rate movements, then US demographic transitions can considerably affect inflation rate dynamics.

I assume that r_t^i equals a smoothed path of the real interest rate along the period under analysis[4]. Next, I compare the simulation path to the effective trajectory for ex-ante real interest rate, nominal interest rate, and inflation. In order to obtain a reasonable measure of inflation expectations for the entire sample I estimated a time-varying first-order autorregression as in Hamilton et al. (2016). Figure 4.6 plots Federal Funds Rate (FFR) and the return of 3-Months Treasury Bills (TB - 3M) in both nominal and ex-ante real terms. All of these variables are characterize by a hump-shaped behaviour from 1950 to 2017. Our model suggests that 1 percentage point of the rise in real interest rate from 1950 to the end-1970s is driven by demographic factors. Specifically, the temporal increase in the share of young workers has generated a reduction in savings and produced an increase in the real interest rate during this period. Moreover, from the beginning of 1980s to the present, demographic factors have contributed to the reduction of 3 percentage points in the real interest rate. For this period, both the reversion in the initial rise in fertility rate and

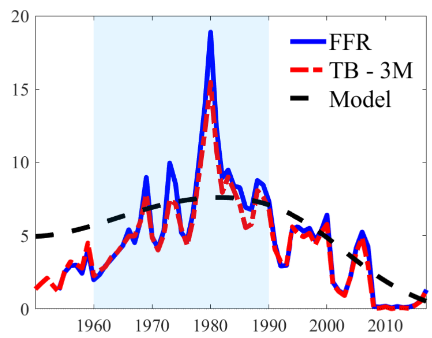
[4]It is roughly constant and equal to its initial steady-state value.

the negative contribution of the increase in life expectancy explain this results. Finally, central bank's miss-perception in the initial rise of the natural interest can partially explain an inflationary episode of 2 percentage points. As long as the natural interest rate declined, a disinflationary episode started. The model suggests that demographics have accounted for 2.5 p.p. of the reduction in the inflation rate since 1980.

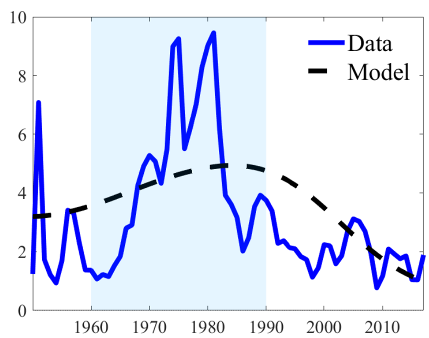
Figure 4.6: Demographics, Real Interest Rate and Inflation



4.6(a): Real Interest Rate



4.6(b): Nominal Interest Rate



4.6(c): Inflation Rate

Note. Shaded areas show 1960-1990 period.

5
Conclusion

I develop an overlapping generations model with life-cycle wage profile, age-dependent mortality rate, liquidity constraints, and nominal rigidities. The model is calibrated to capture the evolution of US demographics and other salient features of the US economy during 1950-2017. Then, I use the model to study the role of demographic trends in explaining real interest rate movements.

There are four main findings in this dissertation. First, I revisit the MPC's heterogeneity channel and highlight it as a powerful transmission mechanism for fertility rate variations. Since young workers are less productive than middle-aged workers and they face liquidity constraints, the marginal propensity to consume is higher for the former. Next, I find that this channel plays a major role in explaining the evolution of the real and natural interest rate. Specifically, I state that the rapid increase in working age population from 1950-1980s have significantly contributed in the rise of real interest rates. The reversion of the fertility process together with the rise in life expectancy triggered a rapid decline in the natural interest rate since 1980s.

Thirdly, given the empirical evidence on large life expectancy forecasting errors, I combine adaptive and eductive learning equilibrium concept and find that the demographic contribution to the rise of the real and natural interest rate can be doubled due to learning-about-longevity process. Finally, since demographic trends affect aggregate saving rates and the natural interest rate, the failure to account for them might trigger inflationary (or deflationary) episodes. Hence, I remark that central banks' miss-perception of the initial rise in the natural interest rate in the 1950s can potentially explain an initial inflationary episode, and, as long as the natural interest rate started declining, a disinflationary process.

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A
Derivation of the model

Here I present a derivation sketch of the non-linear system of partial differential equations which determines equilibrium in the model. First, I transform the agent problem in a stationary problem. Second, I normalize the cross-section distribution functions $g^i()$. Finally, the government budget constraint is expressed in effective-labor units.

A.1 Households

Worker. I adapt the method presented in Duffie and Epstein (1992) for specifying utility processes. With Possion compensated jump process, I conjecture that $U_t^w(j)$ has stochastic differential representation of the form

$$dU_t^w(j) = \mu_t^w(j)dt + [V_t^r(a,j) - U_t^w(j)]\, dJ_t^r(j) + [\underline{V}(a) - U_t^w(j)]\, dJ_t^d(j) \quad \text{(A-1)}$$

where $J_t^d(j)$ and $J_t^r(j)$ are compensated Poisson processes with age-dependent hazard rate. The agent dies (retires) if $J_t^d(j)$ ($J_t^r(j)$) jumps the first time since the agent is born. In the model, the arrival of death (retirement) is time-varying. Following Duffie and Epstein (1992), this implies

$$\mu_t^w(j) = -\left\{u(c_t^w) - \rho U_t^w(j) + \lambda_t^r(j)[V_t^r(a,j) - U_t^w(j)] + \lambda_t^d(j)[\underline{V}(a) - U_t^w(j)]\right\} \quad \text{(A-2)}$$

Since the state evolves according to Equation (2-5) and $dj = dt$, then the solution to this problem is given by a value function $V_t^w(a,j)$ which satisfies Equation (2-6).

Since there is exogenous productivity growth μ_t^x, the HJB equation must be normalized by $X_t^{1-1/\eta}$. Let $\tilde{V}_t^w(a,j) = \frac{V_t^w(a,j)}{X_t^{1-1/\eta}}$ be the intermediate detrended value function and $\hat{Q} = \frac{Q}{X_t}$ denotes the normalized value of variable Q. Thence

$$0 = \max_c \left\{ u\left(\frac{c}{X_t}\right) - \rho\tilde{V}_t^w(a,j) + \partial_a\tilde{V}_t^w(a,j)s_t^w(a,j) + \partial_j\tilde{V}_t^w(a,j) \right.$$
$$\left. + \lambda_t^r(j)[\tilde{V}_t^r(a,j) - \tilde{V}_t^w(a,j)] + \lambda_t^d(j)[\hat{\underline{V}}(\hat{a}) - \tilde{V}_t^w(a,j)] \right.$$

$$+ \partial_t \tilde{V}_t^w(a,j) + \mu_t^x(1-1/\eta)\tilde{V}_t^w(a,j) \Bigg\}$$

where $s_t^w(a,j) = r_t a + (1-\varsigma_t)e(j)w_t + \tilde{e}(j)\Pi_t + \xi_t - \tau_t - c$ and $\underline{\hat{V}}(\hat{a}) = \phi_1(\hat{a}+\phi_2)^{1-1/\eta}$. This HJB equation is still non-stationary because there are permanent changes in w_t. To address it, I characterize the value function in terms of $\hat{a}$ rather than in a itself. Define the final detrended value function $\hat{V}_t^w(\hat{a},j)$ as $\hat{V}_t^w(\hat{a},j) = \tilde{V}_t^w(a,j)$. I will guess that $\hat{V}_t^w(\hat{a},j)$ does not depend on the non-stationary variable X_t and then verify it. It is easy to compute

$$\begin{aligned}
&\partial_a \tilde{V}_t^w(a,j) = \partial_a \hat{V}_t^w\left(\frac{a}{X_t}, j\right) = \frac{1}{X_t}\partial_{\hat{a}}\hat{V}_t^w(\hat{a},j) \\
&\partial_j \tilde{V}_t^w(a,j) = \partial_j \hat{V}_t^w(\hat{a},j) \\
&\partial_t \tilde{V}_t^w(a,j) = \partial_t \hat{V}_t^w\left(\frac{a}{X_t}, j\right) = \partial_t \hat{V}_t^w(\hat{a},j) - \mu_t^x \hat{a}\partial_{\hat{a}}\hat{V}_t^w(\hat{a},j)
\end{aligned}$$

Substituting the intermediate value function by the final one, I obtain the *HJB stationary equation*

$$\begin{aligned}
0 = \max_{\hat{c}} \Bigg\{ & u(\hat{c}) - \rho \hat{V}_t^w(\hat{a},j) + \partial_{\hat{a}}\hat{V}_t^w(\hat{a},j)\hat{s}_t^w(\hat{a},j) + \partial_j \hat{V}_t^w(\hat{a},j) \\
& + \lambda_t^r(j)[\hat{V}_t^r(\hat{a},j) - \hat{V}_t^w(\hat{a},j)] + \lambda_t^d(j)[\underline{\hat{V}}(\hat{a}) - \hat{V}_t^w(\hat{a},j)] \\
& + \partial_t \hat{V}_t^w(\hat{a},j) + \mu_t^x(1-1/\eta)\hat{V}_t^w(\hat{a},j) \Bigg\}
\end{aligned}$$

subject to $\hat{a} \geq -\underline{a}$, where $\hat{s}_t^w(\hat{a},j) = [r_t - \mu_t^x]\hat{a} + (1-\varsigma_t)e(j)\hat{w}_t + \tilde{e}(j)\hat{\Pi}_t + \hat{\xi}_t - \hat{\tau}_t - \hat{c}$. This HJB equation clearly does not depend on X_t. The first order conditions for this problem is

$$\partial_{\hat{a}}\hat{V}_t^w(\hat{a},j) = \partial_{\hat{c}} u(\hat{c}) \tag{A-3}$$

for any $\hat{a}$, j. The state constraint implies a state-constraint boundary condition

$$\partial_{\hat{a}}\hat{V}_t^w(-\underline{a},j) \geq \partial_{\hat{c}} u\left(-[r_t - \mu_t^x]\underline{a} + (1-\varsigma_t)e(j)\hat{w}_t + \tilde{e}(j)\hat{\Pi}_t + \hat{\xi}_t - \hat{\tau}_t\right) \tag{A-4}$$

for any j.

Retirees. Similarly, I can derive the *HJB stationary equation* for retirees

$$0 = \max_{\hat{c}} \Bigg\{ u(\hat{c}) - \rho \hat{V}_t^r(\hat{a},j) + \partial_{\hat{a}}\hat{V}_t^r(\hat{a},j)\hat{s}_t^r(\hat{a},j) + \partial_j \hat{V}_t^r(\hat{a},j)$$

$$+ \lambda_t^d(j)[\underline{\hat{V}}(\hat{a}) - \hat{V}_t^r(\hat{a}, j)] + \partial_t \hat{V}_t^r(\hat{a}, j) + \mu_t^x(1 - 1/\eta)\hat{V}_t^r(\hat{a}, j) \Big\}$$

subject to $\hat{a} \geq -\underline{a}$, where $\hat{s}_t^r(\hat{a}, j) = [r_t - \mu_t^x]\hat{a} + \hat{\xi}_t + \hat{S}_t - \hat{c}$. Hence, the first order condition for retiree's problem is

$$\partial_{\hat{a}} \hat{V}_t^r(\hat{a}, j) = \partial_{\hat{c}} u(\hat{c}) \tag{A-5}$$

for any $\hat{a}$, j. The state constraint implies a state-constraint boundary condition

$$\partial_{\hat{a}} \hat{V}_t^r(-\underline{a}, j) \geq \partial_{\hat{c}} u \left(-[r_t - \mu_t^x]\underline{a} + \hat{\xi}_t + \hat{S}_t \right) \tag{A-6}$$

for any j.

A.2 Kolmogorov Forward Equation

The optimal decision rules of workers and retirees imply optimal drifts for total assets and, together with the exogenous aging process, they induce a joint distribution of wealth and age, $g_t^i(.)$. The evolution of this joint distribution is given by the *Kolmogorov Forward (KF) equation*:

$$\begin{aligned} \partial_t g_t^w(a, j) &= -\partial_a[s_t^w(a, j) g_t^w(a, j)] - \partial_j g_t^w(a, j) - \lambda_t^w(j) g_t^w(a, j) + \delta(j)\delta(a) b_t N_t^w \\ \partial_t g_t^r(a, j) &= -\partial_a[s_t^r(a, j) g_t^r(a, j)] - \partial_j g_t^r(a, j) - \lambda_t^d(j) g_t^r(a, j) + \lambda_t^r(j) g_t^w(a, j) \end{aligned}$$

where s_t^w and s_t^r are the optimal drifts in assets implied by HJB equations, $\lambda_t^w(j)$ equals $\lambda_t^r(j) + \lambda_t^d(j)$, and $\delta(.)$ is the Dirac delta function. Again, I must normalize these distributions. Let $\tilde{g}_t^i(\hat{a}, j)$ be the same measures but in terms of $\hat{a}$ instead of a, then I can directly construct the KF equations for them

$$\begin{aligned} \partial_t \tilde{g}_t^w(\hat{a}, j) &= -\partial_{\hat{a}}[\hat{s}_t^w(\hat{a}, j) \tilde{g}_t^w(\hat{a}, j)] - \partial_j \tilde{g}_t^w(\hat{a}, j) - \lambda_t^w(j) \tilde{g}_t^w(\hat{a}, j) + \delta(j)\delta(\hat{a}) b_t N_t^w \\ \partial_t \tilde{g}_t^r(\hat{a}, j) &= -\partial_{\hat{a}}[\hat{s}_t^r(\hat{a}, j) \tilde{g}_t^r(\hat{a}, j)] - \partial_j \tilde{g}_t^r(\hat{a}, j) - \lambda_t^d(j) \tilde{g}_t^r(\hat{a}, j) + \lambda_t^r(j) \tilde{g}_t^w(\hat{a}, j) \end{aligned}$$

Note that $\int\int g_t^w(a, j) da dj = N_t^w$ plus $\int\int g_t^r(a, j) da dj = N_t^r$ equals N_t. Hence, let $\hat{g}_t^i(\hat{a}, j) = \frac{\tilde{g}_t^i(\hat{a}, j)}{N_t}$ be the normalized measures of workers and retirees which follow

$$\begin{aligned} \partial_t \hat{g}_t^w(\hat{a}, j) &= -\partial_{\hat{a}}[\hat{s}_t^w(\hat{a}, j) \hat{g}_t^w(\hat{a}, j)] - \partial_j \hat{g}_t^w(\hat{a}, j) - [\lambda_t^w(j) + n_t] \hat{g}_t^w(\hat{a}, j) + \delta(j)\delta(\hat{a}) b_t \frac{N_t^w}{N_t} \\ \partial_t \hat{g}_t^r(\hat{a}, j) &= -\partial_{\hat{a}}[\hat{s}_t^r(\hat{a}, j) \hat{g}_t^r(\hat{a}, j)] - \partial_j \hat{g}_t^r(\hat{a}, j) - [\lambda_t^d(j) + n_t] \hat{g}_t^r(\hat{a}, j) + \lambda_t^r(j) \hat{g}_t^w(\hat{a}, j) \end{aligned}$$

A.3 Analysis

Here, I report some analytical results of the model. Specifically, I am interested in both deriving the stationary age distribution of the model and the evolution of it. Computing age-distributions outside the system described in Appendix B is extremely helpful in finding the competitive equilibrium.

Worker Population. Let $N_t^w(j)$ be the number of workers aged j years at t, then worker population grows according to

$$dN_t^w = b_t N_t^w dt - \left[\int_0^\infty \lambda_t^w(j) N_t^w(j) dj\right] dt \tag{A-7}$$

Retiree Population. Similarly, let $N_t^r(j)$ be the number of retirees aged j years at t, henceforth

$$dN_t^r = \left[\int_0^\infty \lambda_t^r(j) N_t^w(j) dj\right] dt - \left[\int_0^\infty \lambda_t^d(j) N_t^r(j) dj\right] dt \tag{A-8}$$

Age Distribution. Contrary to wealth distribution, in the model the age distribution is determined exogenously. These measures only depend on exogenous processes $\{b_t, \lambda_t^r(j), \lambda_t^r(j)\}$ and I make some further analysis for finding the stationary version of them. The main conclusions of this analysis can be summarized in

1. The number of workers at age j can be calculated by

$$N_t^w(j) = b_{t-j} N_{t-j}^w e^{-\int_0^j \lambda_{t-j+v}^w(v) dv} \tag{A-9}$$

   where $\lambda_t^w(j) = \lambda_t^r(j) + \lambda_t^d(j)$.

2. Let $N_t^r(j,s)$ be the number of retirees aged j years with $s \in (0, j]$ years in retirement, hence

$$N_t^r(j,s) = \lambda_{t-s}^r(j-s) N_{t-s}^w(j-s) e^{-\int_0^s \lambda_{t-s+v}^d(j-s+v) dv} \tag{A-10}$$

   Furthermore, $N_t^r(j) = \int_0^j N_t^r(j,s) ds$ and $N_t^r = \int_0^\infty N_t^r(j) dj$.

3. At any stationary equilibrium the measure of workers, retirees, and total population grows at the same rate $n = n^w = n^r$. Moreover, let $S^i(j)$ the survival function for the compensated Poisson process $J^i(j)$, then n is determined by

$$n = b\left[1 - \int_0^\infty \lambda^w(j) S^w(j) e^{-nj} dj\right] \tag{A-11}$$

$$\tag{A-12}$$

4. Finally, the stationary age distribution is

$$\frac{N_t(j)}{N_t} = \frac{N_t(j)}{N_t^w} \Big/ \int_0^\infty \frac{N_t(j)}{N_t^w} dj \tag{A-13}$$

where $\frac{N_t(j)}{N_t^w} = \frac{N_t^w(j)}{N_t^w} + \frac{N_t^r(j)}{N_t^w}$ and

$$\frac{N_t^w(j)}{N_t^w} = be^{-nj} S^w(j) \tag{A-14}$$

$$\frac{N_t^r(j)}{N_t^w} = be^{-nj} S^d(j) \int_0^j \lambda^r(v) \frac{S^w(v)}{S^d(v)} dv \tag{A-15}$$

A.4
Firms and Government

For the remaining of this appendix $\overline{Q} = \frac{Q}{X_t N_t}$ denotes per-capita variables while $\tilde{Q} = \frac{Q}{X_t L_t}$ characterizes variables Q - efficiency labor rate.

Firms. Cost minimization implies

$$\frac{w_t}{X_t} = \alpha m_t \frac{y_{t,i}}{X_t l_{t,i}}$$
$$r_t + \delta = (1-\alpha) m_t \frac{y_{t,i}}{k_{t,i}}$$

Since every firm uses the same capital-efficiency labor ratio, aggregation is simple: $Y_t = K_t^{1-\alpha}[X_t L_t]^\alpha$, then

$$\tilde{Y}_t = \tilde{K}_t^{1-\alpha}$$
$$\hat{w}_t = \alpha m_t \tilde{K}_t^{1-\alpha}$$
$$r_t + \delta = (1-\alpha) m_t \tilde{K}_t^{-\alpha}$$

Note that factor prices depend only on capital-efficiency labor ratio $\tilde{k}_t$. Per-capita variables are given by

$$\overline{Y}_t = \tilde{Y}_t \frac{L_t}{N_t}$$
$$\overline{K}_t = \tilde{K}_t \frac{L_t}{N_t}$$

Furthermore, using $\frac{\dot{\tilde{Y}}_t}{\tilde{Y}_t} = \frac{\dot{\overline{Y}}_t}{\overline{Y}_t} - \frac{\dot{X}_t}{X_t} - \frac{\dot{N}_t}{N_t}$, I can rewrite the NK Phillips Curve in terms of per-capita variables

$$\dot{\pi}_t = \left[r_t + \mu_t^x + n_t - \frac{\dot{\overline{Y}}_t}{\overline{Y}_t} \right] \pi_t - \frac{\varepsilon}{\theta}(m_t - m^*) \tag{A-16}$$

Government. Normalize Equation (2-12) and obtain

$$\dot{\overline{B}}_t = [r_t - \mu_t^x - n_t]\overline{B}_t + \overline{G}_t - \frac{N_t^w}{N_t}\hat{\tau}_t$$

Using fiscal rules: $B_t = b_y Y_t$ and $G_t = g_y Y_t$,

$$\dot{\overline{B}}_t = [(r_t - \mu_t^x - n_t)b_y + g_y]\tilde{K}_t^{1-\alpha}\frac{L_t}{N_t} - \frac{N_t^w}{N_t}\hat{\tau}_t$$

or, equivalently

$$\hat{\tau}_t = \left(\frac{N_t^w}{N_t}\right)^{-1} \left([(r_t - \mu_t^x - n_t)b_y + g_y]\tilde{K}_t^{1-\alpha}\frac{L_t}{N_t} - \dot{\overline{B}}_t\right) \tag{A-17}$$

Finally, social security and bequests can be normalized as follows:

$$\hat{S}_t = \varsigma_t \hat{w}_t \frac{L_t}{N_t^r} \tag{A-18}$$

$$\hat{\xi}_t = \int_0^\infty \lambda_t^d(j) \left\{ \sum_{i \in \{w,r\}} \int \hat{a}\hat{g}_t^i(\hat{a}, j) da \right\} dj \tag{A-19}$$

B
Competitive Equilibrium

$$0 = \max_{\hat{c}} \Big\{ u\left(\hat{c}\right) - \rho \hat{V}^w_t(\hat{a}, j) + \partial_{\hat{a}} \hat{V}^w_t(\hat{a}, j) \hat{s}^w_t(\hat{a}, j) + \partial_j \hat{V}^w_t(\hat{a}, j)$$
$$+ \lambda^r_t(j)[\hat{V}^r_t(\hat{a}, j) - \hat{V}^w_t(\hat{a}, j)] + \lambda^d_t(j)[\underline{\hat{V}}(\hat{a}) - \hat{V}^w_t(\hat{a}, j)]$$
$$+ \mu^x_t(1 - 1/\eta)\hat{V}^w_t(\hat{a}, j) + \partial_t \hat{V}^w_t(\hat{a}, j) \Big\} \tag{B-1}$$

$$0 = \max_{\hat{c}} \Big\{ u\left(\hat{c}\right) - \rho \hat{V}^r_t(\hat{a}, j) + \partial_{\hat{a}} \hat{V}^r_t(\hat{a}, j) \hat{s}^r_t(\hat{a}, j)$$
$$+ \partial_j \hat{V}^r_t(\hat{a}, j) + \lambda^d_t(j)[\underline{\hat{V}}(\hat{a}) - \hat{V}^r_t(\hat{a}, j)]$$
$$+ \mu^x_t(1 - 1/\eta)\hat{V}^r_t(\hat{a}, j) + \partial_t \hat{V}^r_t(\hat{a}, j) \Big\} \tag{B-2}$$

$$0 = -\partial_t \hat{g}^w_t(\hat{a}, j) - \partial_{\hat{a}}[\hat{s}^w_t(\hat{a}, j)\hat{g}^w_t(\hat{a}, j)] - \partial_j \hat{g}^w_t(\hat{a}, j)$$
$$- [\lambda^w_t(j) + n_t]\hat{g}^w_t(\hat{a}, j) + \delta(j)\delta(\hat{a}) b_t \frac{N^w_t}{N_t} \tag{B-3}$$

$$0 = -\partial_t \hat{g}^r_t(\hat{a}, j) - \partial_{\hat{a}}[\hat{s}^r_t(\hat{a}, j)\hat{g}^r_t(\hat{a}, j)] - \partial_j \hat{g}^r_t(\hat{a}, j)$$
$$- [\lambda^d_t(j) + n_t]\hat{g}^r_t(\hat{a}, j) + \lambda^r_t(j)\hat{g}^w_t(\hat{a}, j) \tag{B-4}$$

$$\tilde{Y}_t = \tilde{K}_t^{1-\alpha} \tag{B-5}$$

$$\hat{w}_t = \alpha m_t \tilde{K}_t^{1-\alpha} \tag{B-6}$$

$$r_t = (1-\alpha) m_t \tilde{K}_t^{-\alpha} - \delta \tag{B-7}$$

$$\overline{Y}_t = \tilde{Y}_t \frac{L_t}{N_t} \tag{B-8}$$

$$\overline{K}_t = \tilde{K}_t \frac{L_t}{N_t} \tag{B-9}$$

$$0 = \frac{\varepsilon}{\theta}(m_t - m^*) + \dot{\pi}_t - \left[r_t + \mu^x_t + n_t - \frac{\dot{\overline{Y}}_t}{\overline{Y}_t} \right] \pi_t \tag{B-10}$$

$$\overline{\Pi}_t = \left(1 - m_t - \frac{\theta}{2}\pi_t^2 \right) \tilde{K}_t^{1-\alpha} \frac{L_t}{N_t} \tag{B-11}$$

$$\dot{\overline{B}}_t = [(r_t - \mu^x_t - n_t) b_y + g_y] \tilde{K}_t^{1-\alpha} \frac{L_t}{N_t} - \frac{N^w_t}{N_t} \hat{\tau}_t \tag{B-12}$$

$$\hat{S}_t = \varsigma_t \hat{w}_t \frac{L_t}{N^r_t} \tag{B-13}$$

$$\hat{\xi}_t = \int_0^\infty \lambda_t^d(j) \left\{ \sum_{i \in \{w,r\}} \int \hat{a} \hat{g}_t^i(\hat{a}, j) d\hat{a} \right\} dj \tag{B-14}$$

$$0 = \sum_{i \in \{w,r\}} \int \int \hat{a} \hat{g}_t^i(\hat{a}, j) d\hat{a} dj - \left[\overline{K}_t + \overline{B}_t \right] \tag{B-15}$$

where

$$\begin{aligned} \hat{s}_t^w(\hat{a}, j) &= [r_t - \mu_t^x]\hat{a} + (1 - \varsigma_t) e(j) \hat{w}_t + \tilde{e}(j) \hat{\Pi}_t + \hat{\xi}_t - \hat{\tau}_t - \hat{c}_t^w(a, j) \\ \hat{s}_t^r(\hat{a}, j) &= [r_t - \mu_t^x]\hat{a} + \hat{\xi}_t + \hat{S}_t - \hat{c} \\ \overline{B}_t &= b_y \overline{Y}_t \\ \overline{G}_t &= g_y \overline{Y}_t \end{aligned}$$

C
Numeric Algorithm

I explain the numeric algorithm in two parts. I start explaining how to compute the stationary equilibrium and then how to calculate transition dynamics.

C.1
Stationary Equilibrium

HJB equation. Based on Achdou et al. (2017) and Ahn et al. (2017), I use implicit methods and upwind scheme. Let $\Delta \in \mathbb{R}^+$, then for a given grid in asset and age, $i \in \{1, \dots, I\}$ and $j \in \{1, \dots, J\}$, the HJB equation becomes

$$\begin{aligned}
\frac{v_w^{n+1}(i,j) - v_w^n(i,j)}{\Delta} &= u(c_w^n(i,j)) - \hat{\rho}^w v_w^{n+1}(i,j) + \lambda^r(j) v_r^{n+1}(i,j) + \lambda^d(j)\underline{V}(i) \\
&\quad + \partial_a v_w^{n+1}(i,j) s_w^n(i,j) + \partial_j v_w^{n+1}(i,j) \\
\frac{v_r^{n+1}(i,j) - v_r^n(i,j)}{\Delta} &= u(c_r^n(i,j)) - \hat{\rho}^r v_r^{n+1}(i,j) + \lambda^d(j)\underline{V}(i) \\
&\quad + \partial_a v_w^{n+1}(i,j) s_w^n(i,j) + \partial_j v_w^{n+1}(i,j)
\end{aligned}$$

where $\hat{\rho}^w = \rho + \lambda^w(j) - \mu^x(1 - 1/\eta)$ and $\hat{\rho}^r = \rho + \lambda^d(j) - \mu^x(1 - 1/\eta)$.

Stacking $\{v_r(i,j);\ v_w(i,j)\}$ in the vector $\mathbf{v}$ and $\{c_r(i,j);\ c_w(i,j)\}$ in the vector $\mathbf{c}$

$$\frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta} = \mathbf{u}(\mathbf{c}) + \underline{\mathbf{v}} + \mathbf{x}\mathbf{v}^{n+1} + \mathbf{A}^n\mathbf{v}^{n+1} + \mathbf{C}\mathbf{v}^{n+1}$$

where $\mathbf{x}$ groups $\hat{\rho}^k$, $\mathbf{A}^n$ is related to saving decisions, and C associated to the ageing process. In order to solve the HJB equation by *implicit method* I iterate

$$\mathbf{v}^{n+1} = (1/\Delta - \mathbf{x} - \mathbf{A}^n - \mathbf{C})^{-1}\left[\mathbf{u}(\mathbf{c}) + \underline{\mathbf{v}} + \frac{\mathbf{v}^n}{\Delta}\right] \tag{C-1}$$

until convergence.

KF equation. Let $\mathbf{A} = \lim_{n\to\infty} \mathbf{A}^n$ and $\mathbf{g}$ be the stacked version of $\{g_r(i,j);\ g_w(i,j)\}$, then from FK equation I obtain

$$0 = \mathbf{A}^{\mathrm{T}}\mathbf{g} + \mathbf{B}\mathbf{g} + \mathbf{b}$$

where $\mathbf{B}$ is a matrix related to the ageing process and $\mathbf{b}$ characterizes the Dirac function in KF equation, hence

$$\mathbf{g} = -[\mathbf{A}^{\mathrm{T}} + \mathbf{B}]^{-1}\mathbf{b} \tag{C-2}$$

Algorithm. The algorithm to compute the stationary equilibrium is simple.

1. Guess $\tilde{K}$.
2. Calculate implied prices:

$$\begin{aligned} m^\star &= \frac{\epsilon - 1}{\epsilon} \\ r &= (1-\alpha)m^\star \tilde{K}^{-\alpha} - \delta_k \\ \hat{w} &= \alpha m^\star \tilde{K}^{1-\alpha} \end{aligned}$$

3. Calculate implied profits

$$\overline{\Pi} = (1 - m^\star)\tilde{K}^{1-\alpha} = \frac{1}{\epsilon}\tilde{K}^{1-\alpha}\frac{L}{N}$$

4. Compute fiscal policy

$$\hat{\tau} = [(r - \mu^x - n)b_y + g_y]\tilde{K}^{1-\alpha}\frac{L}{N^w}$$

5. Solve household's problem for both retirees and workers: eq. (C-1) and eq. (C-2).
6. Calculate aggregate asset demand: $\overline{A}$, and the aggregate asset supply:

$$\overline{A}^s = \left(\tilde{K} + b_y\tilde{K}^{1-\alpha}\right)\frac{L}{N}$$

7. Compute excess demand:

$$\Lambda = |\overline{A}^s - \overline{A}|$$

8. If Λ is close to zero the equilibrium have been found. Otherwise update the guess and go back to step 2.

C.2
Transition Dynamics

A similar algorithm can be used the economy's impulse response after an unanticipated (zero probability) shock followed by a deterministic transition.

HJB dynamic equation. Using previous notation, I can solve $\mathbf{v}_t$ using the dynamic HJB equation and a terminal condition $\mathbf{v}_T = \lim_{n\to\infty} \mathbf{v}^n$,

$$\frac{\mathbf{v}_{t-1} - \mathbf{v}_t}{\Delta t} = \mathbf{u}(\mathbf{c_t}) + \underline{\mathbf{v}} + \mathbf{x}_t\mathbf{v}_t + \mathbf{A}_t\mathbf{v}_t + \mathbf{C}\mathbf{v}_t$$

where $\mathbf{x}$ and $\mathbf{A}$ are now time-dependent, thus, from $t = T, \ldots, 1$, I compute

$$\mathbf{v}_{t-1} = (1/\Delta - \mathbf{x}_t - \mathbf{A}_t - \mathbf{C})^{-1}\left[\mathbf{u}(\mathbf{c_t}) + \underline{\mathbf{v}} + \frac{\mathbf{v}_t}{\Delta t}\right] \tag{C-3}$$

KF dynamic equation. Given an initial condition $\mathbf{g}_0 = \mathbf{g}$ and the KF equation, I can solve for $\{\mathbf{g}_t\}$

$$\frac{\mathbf{g}_{t+1} - \mathbf{g}_t}{\Delta t} = \mathbf{A}_t^{\mathrm{T}}\mathbf{g}_{t+1} + \mathbf{B}_t\mathbf{g}_{t+1} + \mathbf{b}_t$$

hence,

$$\mathbf{g}_{t+1} = \left[\mathbf{I} - \Delta t\left(\mathbf{A}_t^{\mathrm{T}} + \mathbf{B}_t\right)\right]^{-1}[\mathbf{g}_t + \Delta t\mathbf{b}_t] \tag{C-4}$$

Algorithm. The algorithm to calculate the transition dynamics is

1. Compute the stationary equilibrium (initial $s0$ and final sf).
2. Guess a path for interest rate, $\{r_t\}_0^T$.
3. Using Taylor equation forwardly in time, $\pi_{-1} = 0$, solve for inflation rate

$$\frac{d\pi_t}{dt} = -\theta^i[i_t - r_t^i - \phi_\pi\pi_t] - \frac{dr_t}{dt}$$

4. Using Phillips Curve backward in time (with $m_T = m^*$) and marginal cost equation I must solve,

$$m_t = m^\star + \frac{\theta}{\epsilon}\left[\left(r_t - (1-\alpha)\frac{\dot{\tilde{K}}_t}{\tilde{K}_t} - \mu_t^x - n_t^w\right)\pi_t - \dot{\pi}_t\right]$$

$$\tilde{K}_t = \left[\frac{r_t + \delta_k}{(1-\alpha)m_t}\right]^{-1/\alpha}$$

5. Compute wages and profits

$$\hat{w}_t = \alpha m_t \tilde{K}_t^{1-\alpha}$$
$$\overline{\Pi}_t = \left(1 - m_t - \frac{\theta}{2}\pi_t^2\right) \tilde{K}_t^{1-\alpha}$$

6. Find fiscal taxes by

$$\overline{B}_t = b_y \tilde{K}_t^{1-\alpha} \frac{L_t}{N_t}$$
$$\hat{\tau}_t = \left(\frac{N_t^w}{N_t}\right)^{-1} \left(\left[(r_t - \mu_t^x - n_t)b_y + g_y\right] \tilde{K}_t^{1-\alpha} \frac{L_t}{N_t} - \dot{\overline{B}}_t\right)$$

7. Solve household's problem using eq. (C-3) and eq. (C-4).

8. Compute aggregate demand $\overline{A}_t$ and supply $\overline{A}$. Calculate excess demand

$$\Lambda_t = |\overline{A}_t^s - \overline{A}_t|$$

9. If, for all t, Λ_t is close to zero, then the equilibrium have been found. Otherwise update the guess and go back to step 2.

D
Adaptive-Eductive Learning Equilibrium - AELE

In this chapter, I explain the learning equilibrium concept used in Section 4.3. It is worth emphasizing that I present it as general as possible, then it can be used in other kind of Heterogeneous Agent (HA) models.[1]

Exogenous System. Let $\mathbf{x}_t$ be a vector of exogenous process and $\mathbf{u}_t$ innovations over it, then assume that I can express the dynamics for this vector as a system of linear differential equations

$$\mathbf{0} = \mathbf{\Phi}_0 \frac{\mathsf{d}\mathbf{x}_t}{\mathsf{d}t} + \mathbf{\Phi}_1 \mathbf{x}_t + \mathbf{u}_t \tag{D-1}$$

where $\mathbf{u}_t$ is a vector of shocks. For instance, in the above model $\mathbf{x}_t$ contains productivity growth and demographic variables and I approximate the functions over a discretized grid of ages $\mathcal{J} = \{j_1 = 0, j_2, \ldots, j_J\}$ in order to obtain the representation of Equation (D-1).

Endogenous System. The endogenous part of any system consist of a vector of value functions $\mathbf{v}_t$, a vector of distributions $\mathbf{g}_t$, a vector of policy instruments $\mathbf{h}_t$, and a vector of prices $\mathbf{p}_t$. Thus, the endogenous system are determined by:

$$\text{[HJB equations]} \quad \mathbf{0} = \mathbf{U}(\mathbf{v_t}) + \mathbf{A}(\mathbf{v_t}; \mathbf{p_t}, \mathbf{h_t}, \mathbf{x_t}) + \frac{\mathsf{d}\mathbf{v_t}}{\mathsf{d}t} \tag{D-2}$$

$$\text{[KF equations]} \quad \frac{\mathsf{d}\mathbf{g_t}}{\mathsf{d}t} = \mathbf{G}(\mathbf{v_t}; \mathbf{p_t}, \mathbf{h_t}, \mathbf{x_t})\mathbf{g_t} \tag{D-3}$$

$$\text{[Policy equations]} \quad \frac{\mathsf{d}\mathbf{h}_t}{\mathsf{d}t} = \mathbf{H}(\mathbf{g_t}; \mathbf{p_t}, \mathbf{h_t}, \mathbf{x_t}) \tag{D-4}$$

$$\text{[Clearing Market Cond.]} \quad \mathbf{p_t} = \mathbf{P}(\mathbf{g_t}; \mathbf{h_t}, \mathbf{x_t}) \tag{D-5}$$

I also approximate the value function and distribution over a discretized grid of asset holdings $\mathcal{A} = \{a_1 = -\underline{a}, a_2, \ldots, a_I\}$. Hence, denote the value function and distribution along the discrete state-space $\mathcal{A} \times \mathcal{J}$ using vectors $\mathbf{v}_t = [(V_t^w(a,j))_{a\in\mathcal{A}, j\in\mathcal{J}}, (V_t^r(a,j))_{a\in\mathcal{A}, j\in\mathcal{J}}]'$ and

[1]Recent studies introduce bounded rationality in HA models to examine the role of Monetary Policy within this framework. See Farhi and Werning (2017), Qiu (2018), and Molavi (2019).

$\mathbf{g}_t = [(g_t^w(a,j))_{a\in\mathcal{A},j\in\mathcal{J}}, (g_t^r(a,j))_{a\in\mathcal{A},j\in\mathcal{J}}]'$.

Point-wise Constrained Foresight Equilibrium (PCFE). Taking $\{\mathbf{x}_{t,s}^e\}_{s\geq t}$ with $\mathbf{x}_{t,t}^e = \mathbf{x}_t$ and $\{\mathbf{g}_s, \mathbf{p}_s, \mathbf{h}_s\}_{s<t}$, the *point-wise constrained foresight equilibrium* (PCFE) is given by the solution $\{\mathbf{v}_{t,s}, \mathbf{g}_{t,s}, \mathbf{p}_{t,s}, \mathbf{h}_{t,s}\}_{s\geq t}$ of the following system:

$$\mathbf{0} = \mathbf{U}(\mathbf{v_{t,s}}) + \mathbf{A}(\mathbf{v_{t,s}}; \mathbf{p_{t,s}}, \mathbf{h_{t,s}}, \mathbf{x^e_{t,s}}) + \frac{\mathsf{d}\mathbf{v_{t,s}}}{\mathsf{d}s} \tag{D-6}$$

$$\frac{\mathsf{d}\mathbf{g_{t,s}}}{\mathsf{d}s} = \mathbf{G}(\mathbf{v_{t,s}}; \mathbf{p_{t,s}}, \mathbf{h_{t,s}}, \mathbf{x^e_{t,s}})\mathbf{g_{t,s}} \tag{D-7}$$

$$\frac{\mathsf{d}\mathbf{h}_{t,s}}{\mathsf{d}s} = \mathbf{H}(\mathbf{g_{t,s}}; \mathbf{p_{t,s}}, \mathbf{h_{t,s}}, \mathbf{x^e_{t,s}}) \tag{D-8}$$

$$\mathbf{p_{t,s}} = \mathbf{P}(\mathbf{g_{t,s}}; \mathbf{h_{t,s}}, \mathbf{x^e_{t,s}}) \tag{D-9}$$

Adaptive-Eductive Learning Equilibrium (AELE). The equilibrium concept used in Section 4.3 combines **adaptive or econometric learning equilibrium**[2] (about exogenous system) and PCFE (for the endogenous system).

First, I assume that expectations about exogenous vector $\mathbf{x}_t$ are recursively updated as in Least-Square Econometric Learning models. In this environment, agents know the entire path of exogenous shocks $\{\mathbf{u}_t\}_{t\geq 0}$ but not the parameters governing the Data Generator Process (DGP). That is, they should learn about $\{\mathbf{\Phi}_0, \mathbf{\Phi}_1\}$. Then, given $\{\mathbf{x}_s\}_{s\leq t}$, they estimate $\{\mathbf{\Phi}_0, \mathbf{\Phi}_1\}$ and forecast $\mathbf{x}$ based on their Perceived Law of Motion (PLM),

$$\mathbf{0} = \hat{\mathbf{\Phi}}_{0,t}\frac{\mathsf{d}\mathbf{x}^e_{t,s}}{\mathsf{d}s} + \hat{\mathbf{\Phi}}_{1,t}\mathbf{x}^e_{t,s} + \mathbf{u}_s \tag{D-10}$$

for all $s > t$. In this document, I use a traditional updating rule based on a constant Kalman gain $\mathbf{\Gamma_i}$

$$\mathsf{d}\hat{\mathbf{\Phi}}_{i,t} = -\mathbf{\Gamma_i}[\hat{\mathbf{\Phi}}_{i,t} - \mathbf{\Phi}_{i,t}]\mathsf{d}t \tag{D-11}$$

for $i \in \{1, 2\}$.

Secondly, at any moment and given expectations on exogenous variables $\{\mathbf{x}^e_{t,s}\}_{s>t}$ at t, expectations about $\{\mathbf{g}_t,\ \mathbf{p}_t,\ \mathbf{h}_t\}$ are formed eductively. Hence, I define **Adaptive-Eductive Learning Equilibrium (AELE)** concept as a continuous sequence of PCFE $\{\{\mathbf{v}_{t,s}, \mathbf{g}_{t,s}, \mathbf{p}_{t,s}, \mathbf{h}_{t,s}, \mathbf{n}_{t,s}\}_{s\geq t}\}_{t\geq 0}$ consistent with Least-Square learning expectations $\{\mathbf{x}^e_{t,s}\}_{s\geq t}$ with $\mathbf{x}^e_{t,t} = \mathbf{x}_t$ for every t.

[2]See for example Evans and Honkapohja (2001), Evans and Honkapohja (2009), and Eusepi and Preston (2011).

Redefining Perfect Foresight Equilibrium (PFE). PFE assumes that agents know the trajectory of the unexpected shock $\{\mathbf{u}_t\}$ and the DGP for exogenous process, i.e., $\hat{\mathbf{\Phi}}_{1,t} = \mathbf{\Phi}_1$ and $\hat{\mathbf{\Phi}}_{0,t} = \mathbf{\Phi}_0$ for all t. Hence,the PFE can be redefined as a sequence of PCFE $\{\{\mathbf{v}_{t,s}, \mathbf{g}_{t,s}, \mathbf{p}_{t,s}, \mathbf{h}_{t,s}, \mathbf{n}_{t,s}\}_{s\geq t}\}_{t\geq 0}$ consistent with true exogenous DGP process $\{\mathbf{x}^e_{t,s} = \mathbf{x}_s\}_{s\geq t}$ for every t.