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Abstract. We propose a specification of a general equilibrium model with securitization of collateral-backed promises and discuss the role of physical collateral to avoid, in equilibrium, pessimistic beliefs about the future rates of default. Promises are pooled in either pass-through securities or collateralized loans obligations (CLO), allowing the existence of different seniority levels among tranches in the same CLO. In case of default, borrowers may also suffer extra-economic penalties, which are internalized into a structure of non-ordered preferences. In this context, we prove the existence of an equilibrium in that investors are not over-pessimistic about the payments of derivatives.

Keywords. Asset Backed-Securitization, Extra-economic Default Penalties, Collateral, Non-ordered Preferences.
JEL Classification: D52, D91.

1. Introduction

In financial markets, securitization of debt contracts has been a mean for financial institutions to reduce risk in their balance sheets. Moreover, it allows better portfolio diversification, as investors have access to a broader pool of assets. In this sense, securitization has an important role in improving efficiency.

Assets that are issued to diversify the credit risk are traditionally called Asset Backed Securities (ABS), when protected by a pool of loans or receivables, and Mortgage Backed Securities (MBS), when backed by residential or commercial mortgage loans. However, a consensus does not exist about the differentiation of the two classes.[1] For simplicity, we consider the Mortgage Backed Securities as particular types of ABS.

From the risk distribution perspective, there are two ways a given poll of assets (e.g. loans, mortgage, receivables, bonds) can be securitized into a family of ABS: 1) allowing the amount of default to be divided pro-rata among the different ABS; or 2) allowing senior-subordinated structures

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[1] In the United States, ABS and MBS are considered different types of securities. In some European countries, MBS are a particular type of ABS. For a detailed description of securitization practice and the characteristics of different markets, see Tavakoli [15] or Hayre [12].

among the different derivatives at the moment of payments. When payments are distributed pro-rata, ABS are also called Pass-through securities. On the other hand, when there is a senior-subordinated structure among the ABS, they are called Collateralized Loan Obligations (CLO) or tranches. When the underlying loans are mortgages, we could also refer to CLO as Collateralized Mortgage Obligations (CMO).

Our objective is to insert these structures in a general equilibrium model, generalizing the seminal works that extend the traditional General Equilibrium Model to allow for credit risk, collateral and extra-economic penalties (see Geanakoplos [10], Geanakoplos and Zame [11] and Dubey, Geanakoplos and Shubik [7]). Moreover, we are interested in studying the role of physical collateral requirements to avoid excess of investor pessimism about the future rates of default.

Our economy has two time periods and there is uncertainty about the state of nature in the second period. Commodities may be durable, assets are real, and a finite number of agents, which are characterized by non-ordered preferences, can trade on spot markets. There are two types of financial contracts available in these markets: (i) primitive securities, that are sold by the borrowers and are backed by physical collateral requirements, that can depend of the prices; and (ii) derivatives, which are bought by the lenders and are backed by classes of primitives. In the case of default, borrowers are burdened by both the seizure of collateral requirements and by extra-economic penalties, which are incorporated into our structure of preferences. These extra-economic penalties reflect the existence of legal or moral enforcements and may differ among agents.

We suppose that financial intermediaries, whose role is limited to pool individual claims, buy the debts and issue derivatives. These ABS may be (i) families of tranches, which receive payments following a senior-subordinated structure, guaranteeing that tranches with lower priority receive nothing unless those with higher priority are fully paid; or (ii) Pass-through securities. The value of aggregate derivatives purchases must match, at equilibrium, the value of short sales on primitives. Moreover, as financial markets are anonymous, lenders take the rates of payment of the derivatives as given. At equilibrium, these rates are determined in such manner that the total value of deliveries matches the total value of payments.

Given this context, suppose that (i) prices of primitive assets are zero, and (ii) lenders are over-pessimistic, in the sense that they expect to suffer total default at each state of nature. Then agents can not borrow anything at the first period, and lenders can not expect to receive any payment at the second period. Therefore, assets are not traded and a pure spot market equilibrium always exists. However, assets are protected by physical collateral requirements and goods are durable. Thus, it is not rational that lenders expect to suffer total default in a derivative, whenever both the depreciated collateral and the original promises of their primitives have positive values.

Hence, using the fact that promises are backed by physical collateral requirements, we propose and show the existence of an equilibrium refinement, called *non-trivial equilibrium*, in which over-pessimistic beliefs are ruled out. A *non-trivial* equilibrium will be an equilibrium in which agents can expect to suffer total default, in a given state of nature, only on derivatives that are backed by primitives that have either zero real promises or zero depreciated bundle of collateral requirements.

1.1. **Insertion in Literature and Contributions.** The study of securitization structures in a general equilibrium context, in which agents can default in their promises, has experienced an increasing importance over the last few years. Geanakoplos [10] and Geanakoplos and Zame [11] studied the existence of an equilibrium in models in which borrowers are burdened by collateral requirements in order to protect lenders from credit risk. In these models of collateralized loans, the only enforcement in case of default is the seizure of the collateral. Therefore, borrowers make strategic default and trade directly with lenders, that expect to receive the minimum between the depreciated value of the collateral and the value of the original promises. In this context, equilibrium existence follows from the scarceness of physical collateral requirements, which guarantees that short sales are bounded at equilibrium.

Extensions of these models, allowing endogenous collateral requirements, were made recently by Araujo, Fajardo and Páscoa [2] and Martins-da-Rocha and Torres-Martínez [13]. In order to overcome the non convexity induced by assets returns, in a context in which borrowers can choose their collateral guarantees, Araujo, Fajardo and Páscoa [2] suppose that there exists a continuum of agents that trade assets. Financial intermediaries pool the debts of borrowers associated with a given promise and issue a single derivative security, which is bought by the lenders. Investors expect to receive, at each state of nature in the future, the mean value of the payments made by the debtors. In a similar context, but with a finite number of agent types, Martins-da-Rocha and Torres-Martínez [13] neutralize the non-convexity on asset returns allowing lenders to raise capital by securitizing part of these assets. In both models, equilibrium with endogenous collateral always exists.

Our financial structure differs from their models in at least two aspects: (i) in our model, financial intermediaries can issue more than one derivative, allowing Pass-through derivatives and senior-subordinated structures; (ii) although we allow collateral requirements that can depend on prices, borrowers can not choose directly their guarantees.

Other types of models allow default without physical collateral requirements, but they burden borrowers by extra-economic penalties, proportional to the real value of default. In this context, Zame [16] studies the advantages of default in order to promote efficiency and Dubey, Geanakoplos and Shubik [7] prove the existence of an equilibrium in a two-period model with incomplete markets.

As we do not assume the existence of utility functions, the linear utility penalties used in the articles above do not make sense in our framework. Therefore, we allow extra-economic enforcements in a broader sense, internalizing the default punishments into the non-ordered preferences. As a particular case, we obtain the representation of preferences used by Dubey, Geanakoplos and Shubik [7].

Furthermore, models in which agents take payment rates of assets as given can have a *pure spot market equilibrium*, whose existence can be easily proved. As explained before, this problem comes from very pessimistic beliefs of lenders about future rates of payment. In order to avoid this pathology and to allow assets to be traded at equilibrium, it is interesting to refine this equilibrium concept.

In this direction, Dubey, Geanakoplos and Shubik [7] propose a refinement concept as a limit of abstract ε-boosted equilibriums, in which an abstract agent (that can be interpreted as a government) buys and sells ε units of each asset and gives no default at each state of nature. Hence, for a fixed ε, lenders are not very pessimistic, as they perfectly foresee strictly positive payments for each asset. By taking the limit, when ε goes to zero, they obtain a refined equilibrium.

In our model, as primitives are backed by physical bundles, we can introduce another refinement concept using the fact that primitive assets deliver, in case of default, at least the depreciated value of the collateral. Thus, as explained before, lenders expect to receive a positive payment when the physical collateral, associated to the underlying primitives, does not disappear from the economy. Although we can not guarantee that assets are traded at equilibrium, our refinement concept assures that the absence of negotiation, when it is the case, is not a consequence of over-pessimistic beliefs.

The paper is organized as follows: Section 2 describes the model; in Section 3, we state the definition of *Equilibrium with Asset-Backed Securitization*; in Section 4 we discuss the role of collateral to avoid over-pessimistic beliefs, and state our refinement concept; and Section 5 is devoted to analyze the assumptions and to state our main result about existence. After some concluding remarks, we make the proof of equilibrium existence in the Appendix.

2. Model

We consider a two-period economy in which agents have uncertainty about the future state of nature. Time periods are denoted by $t \in \{0,1\}$ and we suppose that at the first period, $t=0$, there is no uncertainty (i.e., only one state of nature, denoted by $s=0$, is reached). At the second period, $t=1$, a state of nature is revealed among a finite number of possibilities, $s \in S$. For convenience of notation, we put $S^* = \{0\} \cup S$ to denote the set of states of nature in the economy.

At each state $s \in S^*$ a finite number of perfect divisible commodities $l \in L$ are negotiated in spot markets. These goods can be durable at the first period and, as in Geanakoplos [10] and Geanakoplos and Zame [11], they may suffer depreciation contingent to the state of nature at the second period. This structure, given by linear transformations that are represented by matrixes $Y_s \in \mathbb{R}_+^{L\times L}$, guarantees that when an agent chooses a bundle x at $t=0$, he expects to receive a bundle $Y_s\,x$ if the state of nature $s \in S$ is reached.[2] Note that this structure is general enough to allow perishable goods and perfect durable goods as particular cases.

[2]With more notation, we could have personalized depreciation functions. Moreover, we could also have a depreciation structure characterized by concave functions and we would still be able to guarantee equilibrium. However, as we let the seizure of the depreciated physical collateral requirements be a mechanism of enforcement of the promises payments, it would not be clear what would be the depreciated collateral bundle when a borrower consume more than the required collateral. Take for example the following case: There exists only one good and its depreciation is given by $Y_s(x) = \sqrt{x}$ for all states of nature. If a borrower is obligated to constitute 1 unit of collateral and he decides to consume a total of 4 units of the good, we have that his depreciated bundle at each state is 2 units of the good. What would be his depreciated collateral bundle? $Y_s(1)=1$, $\frac{Y_s(4)}{4}=\frac{1}{2}$ or $Y_s(4)-Y_s(3)=2-\sqrt{3}\cong 0.27$. All being perfctly reasonable answers, but different from each other. On the other hand, with linear depreciation functions the three cases above would be equal.

Commodities in L are traded, at state $s \in S^*$, at prices $p_s \in \mathbb{R}^L_+$. We will denote the commodity price process as $p = (p_s)_{s=0}^S$ and, as usual, we suppose that all physical goods are in positive net supply, that is, there exists physical endowments $W_s \in \mathbb{R}^L_{++}$, at each state $s \in S^*$.

A finite number of agents, $h \in H$, trades commodities at every state, choosing consumption allocations in the commodity space $X_s = \mathbb{R}^L_+$. The space of consumption in the economy is given by $X = \Pi_{s\in S^*} X_s$. Moreover, as in Gale and Mas-Collel [8], at each state of nature the agents receive an initial nominal income given by functions $m^h_s(p_s) \geq 0$. Note that if we suppose that agents receive, at each state $s \in S^*$, a physical endowment $\omega^h_s \in \mathbb{R}^L_+$, the usual framework of general equilibrium models, $m^h_s(p_s) = p_s\omega^h_s$, is a particular case of this structure of income functions. We use this general structure to overcome the survival assumption, i.e. $\omega^h = (\omega^h_s)_{s\in S} \gg 0$ for all $h \in H$.[3]

In our economy, we consider a financial structure in which assets are subject to credit risk. We allow borrowers to negotiate real securities, called primitive assets, which are subject to default and backed by *physical* collateral requirements, which can depend on the price level. On the other side, financial intermediaries, which are limited to pool individual claims, make an asset-backed securitization of these debts contracts, selling derivatives to the lenders.

To give credit risk protection, it is sufficient to impose mechanisms on the financial structure of original loans, in order to burden borrowers in case of default. On this direction, we suppose that enforcements mechanisms, as the seizure of the physical collateral requirements or the punishment via extra-economic penalties, are allowed.

We divide primitive assets into different classes and, for a given class of these debts contracts, we allow the existence of a finite number of derivatives. These derivatives can be of two types: (i) *pass-through securities* (i.e. the payments made by the original promises are distributed pro rata among the derivatives) and (ii) *collateralized loans obligations* (CLO), also called *tranches*, (i.e. securities that have associated an exogenous seniority structure, establishing an order in which derivatives should be payed). For sake of simplicity, only one type of derivative is associated with a given class of primitive assets. In equilibrium, analogous to Araujo, Fajardo and Páscoa [2], the value of the aggregate short sales must match the value of its derivatives aggregate purchase within each family of derivatives. In this sense, our model does not allow, at equilibrium, *overcollateralization* of asset-backed securities.

Remark 1. Although, in real financial markets Collateralized Loan Obligations have an important role protecting lenders from *prepayment risk*, in our model the main goal of this structure is to give lenders more protection from default risk as we work with a two-period model.

In a multi-period model, prepayment risk appears when borrowers have an incentive to pay their promises before the period that was established when they took the loan. However, as derivative promises are state contingent, the amount received as a prepayment need to be reinvested by the issuer of the derivatives in order to pay the future commitments. If financial intermediaries do not

[3] One way of seeing this is considering another example of income function: $m^h_s(p_s) = p_s\omega^h_s(p_s)$, where we let the physical endowment to be a function of prices p_s. In this setting we can replace the assumption $\omega^h_s \gg 0$ for $p_s\omega^h_s(p_s) > 0$ whenever $p_s \neq 0$. This allows that, for each price p_s, $\omega^h_s(p_s)$ do not need to be interior.

have others investment opportunities available, which would give enough return to pay in full the derivatives, the prepayments made by the borrowers can generate default to the investors, without the existence of any default in the primitives.

Formally, a finite number of primitive assets $k \in K$ can be sold at the first period for an unitary price $q_k \in \mathbb{R}_+$. These assets make real promises $A_{s,k} \in \mathbb{R}^L_+$ at each state of nature $s \in S$. Thus, when an agent h sells φ^h_k units of the primitive k, he pays an amount $q_k\varphi^h_k$ and he is burden to constitute a personalized bundle $C_k(p_0, q_k)\varphi^h_k$. The function $C_k : \mathbb{R}^L_+ \times \mathbb{R}_+ \to \mathbb{R}^L_+$ denotes the price-dependent rule to constitute the unitary collateral requirement on asset k, which all agents take as given.[4]

Our approach includes the case considered by Geanakoplos and Zame [11], $C_k(p_0, q_k) = \overline{C}_k$, in which the collateral requirements do not depend on the price level. Moreover, as in Araujo, Páscoa and Torres-Martínez [3], we can consider the case in which, except for some upper and lower bounds, the value of the collateral requirements maintain a fixed margin f over the asset price, where the margin of the collateral requirements is given by the ratio between the value of collateral and the asset price, $\frac{p_0 C_k}{q_k}$,

$$C_k(p_0, q_k)_1 = \min\left\{\overline{f}, \max\left\{\frac{q_k f}{p_{o,1}}, \underline{f}\right\}\right\},$$
$$C_k(p_0, q_k)_l = 0, \ \ \forall\, l \neq 1.$$

Furthermore, we suppose that, at each state $s \in S$, the agents can be burdened not only by the seizure of the depreciated bundle of collateral, but also by other non-economic mechanisms, which are incorporated in their preferences. These mechanisms, analogous to utility penalties in Dubey, Geanakoplos and Shubik [7], can induce the agents to pay more than the collateral value at $t = 1$. Hence, an agent h, who borrows φ^h_k units of k, delivers, at each state $s \in S$, a non-negative amount $\delta^h_{s,k}$, which is chosen jointly with the portfolio and consumption allocations and satisfy $\delta^h_{s,k} \geq \min\{p_s A_{s,k};\, p_s Y_s C_k(p_0, q_k)\}\, \varphi^h_k$.

In real markets, financial institutions use both primitive promises and pass-through securities as collateral for CLO. For sake of simplicity, we suppose that CLO are backed only by the original promises. Therefore, we suppose that primitives in K are divided into two disjoint sets $\mathcal{A}_P$ and $\mathcal{A}_C$. Promises in $\mathcal{A}_P$ will back pass-through securities while promises in $\mathcal{A}_C$ will secure collateralized loan obligations. Moreover, families of securities are backed by classes of primitives. Thus, we suppose that the sets $\mathcal{A}_i$, $i \in \{P, C\}$, are partitioned, exogenously, into ζ_i disjoint classes $\mathbb{A}^g_i \subset \mathcal{A}_i$, $i \in \{P, C\}$ and $g \in \{1, 2, \ldots, \zeta_i\}$. For notation convenience, when there is no possibility of mistakes, we refer to a generic class $\mathbb{A}^g_i$ as $\mathbb{A}_i$.

The promises within each class $\mathbb{A}_i$, $i \in \{P, C\}$, are pooled by a financial intermediary that issues a finite collection, $J(\mathbb{A}_i)$, of short-lived (asset-backed) real assets, denoted by $j \in J(\mathbb{A}_i)$. For notation

[4] Although, in their model of exogenous collateral, Geanakoplos and Zame [11] allow lenders to hold part of the collateral requirements, in our case it is not clear what would be the rule for distributing collateral bundles among investors, because we allow a pool of primitive promises. Thus, we suppose that the physical guarantees are both constituted and held by debtors.

convenience, we will denote by J the collection of all derivatives that can be traded on the markets and by $n(\mathbb{A}_i) = \#J(\mathbb{A}_i)$ the number of derivatives associated to the class of primitives $\mathbb{A}_i$.

Pass-through Securities: Given a class $\mathbb{A}_P \subset \mathcal{A}_P$, each derivative $j \in J(\mathbb{A}_P)$ makes individual real promises $A_{s,j} \in \mathbb{R}^L_+$ at each state $s \in S$, and can be bought at prices q_j at the first period. There are no priorities among the different claims $j \in J(\mathbb{A}_P)$ and, therefore, each Pass-through receives a *pro rata* share of the total deliveries made by the primitive assets $k \in \mathbb{A}_P$.

As markets are anonymous (i.e. lenders do not know the identity of the borrowers), agents expect to receive, for each traded unit of the asset $j \in J(\mathbb{A}_P)$, a percentage $r_{s,j}$ of the promises $A_{s,j}$. However, as agents know that derivatives in $J(\mathbb{A}_P)$ are pass-through securities, they expect identical anonymous payment rates for each of them, that is, $r_{s,j} = r_{s,j'}$ for all derivatives j and j' in $J(\mathbb{A}_P)$. For sake of simplicity, we denote the payment rate, common to all the derivatives in a family $J(\mathbb{A}_P)$, as $r_{s,\mathbb{A}_P}$. Thus, if an agent h buys θ^h_j units of $j \in J(\mathbb{A}_P)$, for some $\mathbb{A}_P \subset \mathcal{A}_P$, he pays an amount $q_j\,\theta^h_j$ and expects to receive, at each state $s \in S$, an amount $r_{s,\mathbb{A}_P}\, p_s A_{s,j}\, \theta^h_j$.

Collateralized Loan Obligations: Given a class $\mathbb{A}_C \subset \mathcal{A}_C$, the family of derivatives $J(\mathbb{A}_C)$ is given by $J(\mathbb{A}_C) := \{j^1(\mathbb{A}_C), j^2(\mathbb{A}_C), \ldots, j^{n(\mathbb{A}_C)}(\mathbb{A}_C)\}$, where the CLO $j^m(\mathbb{A}_C)$ has priority over the assets $(j^r(\mathbb{A}_C))_{r>m}$ in relation to promise payments. Analogous to Pass-through securities, each tranche $j \in J(\mathbb{A}_C)$ makes individual real promises $A_{s,j} \in \mathbb{R}^L_+$ at every state $s \in S$ and can be bought at prices q_j at the first period.

Now, as lenders know the securitization structure (i.e., they know what the priorities are among assets in the same family), but markets are anonymous, they expect to receive for each traded unit of the asset $j \in J(\mathbb{A}_C)$ a percentage of the original promises, given by anonymous payment rates $r_{s,j}$. As tranches with lower priority levels suffer default before those with higher priority levels, if a tranche $j^m(\mathbb{A}_C)$ pays in full at state $s \in S$, $r_{s,j^m(\mathbb{A}_C)} = 1$, then all the derivatives $j^{m'}(\mathbb{A}_C)$, with $m' < m$, pay in full too, $r_{s,j^{m'}(\mathbb{A}_C)} = 1$. Moreover, if an asset $j^m(\mathbb{A}_C)$ gives a partial default, $r_{s,j^m(\mathbb{A}_C)} \in (0,1)$, then all the tranches with higher priority over it pay in full (i.e. $r_{s,j^{m'}(\mathbb{A}_C)} = 1$, for $m' < m$) and all the derivatives that are subordinated to $j^m(\mathbb{A}_C)$ give total default (i.e. $r_{s,j^{m'}(\mathbb{A}_C)} = 0$, for $m' > m$). Therefore, we can suppose that the anonymous payment rates, for the derivatives in the class $J(\mathbb{A}_C)$, belong, at each state of nature, to the *non-convex* set

$$\mathcal{R}(\mathbb{A}_C) := \left\{ r = (r_m) \in [0,1]^{n(\mathbb{A}_C)} : \exists m,\, 1 \le m \le n(\mathbb{A}_C),\, \big(r_{m'} = 1,\, \forall m' < m\big) \wedge \big(r_{m'} = 0,\, \forall m' > m\big) \right\}.$$

It is important to remark that the anonymous payment rates, for both pass-through securities and CLO, are taken as given by the agents, but in equilibrium they are determined in such way that, at each node, the total value of the deliveries will be equal to the total payments.

As it was mentioned above, each agent $h \in H$ is characterized by preferences that may depend on the real amount of default. Formally, denoting the total real default made by an agent, in each primitive asset and at each state, by $d = (d_{s,k})_{(s,k)\in S\times K} \in \mathbb{R}^{S\times K}_+$, we suppose that for each h there exists a correspondence $Q^h : X \times \mathbb{R}_+^{K\times S} \twoheadrightarrow X \times \mathbb{R}_+^{K\times S}$, that represents the agent's preferences over consumption bundles and amounts of default, in the sense that $Q^h(x,d)$ denotes the collection of plans (x',d') that are *strictly preferred* to (x,d) by agent h. Note that with this characterization of agents preferences we do not need to assume completeness, transitivity or continuity.

REMARK 2. We have some interesting particular cases. First, given a function $U^h : \mathbb{R}_+^{L\times S^*} \to \mathbb{R}_+$, for a given collection of numbers $\lambda_{s,k}^h \in \mathbb{R}_+$, we can put

$$Q^h(x,d) \equiv \left\{ (x',d') \,:\, U^h(x') - \sum_{s\in S}\sum_{k\in K} \lambda_{s,k}^h d'_{s,k} > U^h(x) - \sum_{s\in S}\sum_{k\in K} \lambda_{s,k}^h d_{s,k} \right\},$$

to recover the representation of preferences used by Dubey, Geanakoplos and Shubik [7], in which an agent h feels a utility level of consumption given by U^h and is burdened by utility penalties proportional to the real amount of default. Moreover, given a strictly monotonic set function $\Omega^h : X \twoheadrightarrow X$, we can define the individual preferences $Q^h(x,d) := \Omega^h(x) \times \mathbb{R}_+^{K\times S}$ in order to have a representation of preferences (possibly non-ordered) in a model in which the only enforcement in case of default is given by the seizure of collateral guarantees. Of course, the traditional analytic representation of preferences by utility functions, as in Geanakoplos and Zame (2002), is a particular case putting $\Omega^h(x) := \left\{ x' \in X \,:\, U^h(x') > U^h(x) \right\}$.

Finally, as agents are price takers, given commodity prices $p = (p_s)_{s\in S^*}$, a price vector for both primitive and derivative assets $q = (q_k, q_j)_{k\in K, j\in J}$, and anonymous payment rates for the derivatives $r = (r_{s,j})_{(s,j)\in S\times J}$, an agent $h \in H$ can choose consumption-financial allocations $\left[x_0^h, x_s^h, \varphi_k^h, \delta_{s,k}^h, \theta_j^h\right]_{(s,k)\in S\times K,\, j\in J}$ subject to,

- First period budget constraint,

$$p_0 x_0^h + \sum_{i\in\{P,C\}} \left[\sum_{\mathbb{A}_i \subset \mathcal{A}_i} \left(\sum_{j\in J(\mathbb{A}_i)} q_j \theta_j^h - \sum_{k\in\mathbb{A}_i} q_k \varphi_k^h \right) \right] \leq m_0^h(p_0), \tag{1}$$

- Collateral requirements constraint,

$$x_0^h \geq \sum_{k\in K} C_k(p_0, q_k) \varphi_k^h, \tag{2}$$

- Payments constraints,

$$\delta_{s,k}^h \geq \min\left\{ p_s A_{s,k}; p_s Y_s C_k(p_0,q_k) \right\} \varphi_k^h, \quad \forall (s,k) \in S\times K. \tag{3}$$

- Second period, state by state, budget constraints,

$$p_s x_s^h \leq m_s^h(p_s) + p_s Y_s x_o^h + \sum_{i\in\{P,C\}} \left[\sum_{\mathbb{A}_i\subset\mathcal{A}_i} \left(\sum_{j\in J(\mathbb{A}_i)} r_{s,j} p_s A_{s,j} \theta_j^h - \sum_{k\in\mathbb{A}_i} \delta_{s,k}^h \right) \right], \ \forall s\in S. \tag{4}$$

When commodity-financial prices are (p,q), and rates of payment are r, the budget set of the agent $h\in H$, denoted by $B^h(p,q,r)$, is given by the collection of consumption-financial allocations

$(x^h, \varphi^h, \delta^h, \theta^h)$ that satisfies equations (1) to (4).

It follows from the considerations above that our *Economy with Asset Backed Securitization* $\mathcal{E}(S^*, \mathcal{H}, \mathcal{L}, \mathcal{F})$ is characterized by the set of all states of nature $S^* = \{0\} \cup S$, the set of agents characteristics $\mathcal{H} = (X, Q^h, m^h)_{h \in H}$, the physical market structure $\mathcal{L} = (L, (Y_s)_{s \in S}, (W_s)_{s \in S^*})$ and the financial structure $\mathcal{F} = \left[K, J, \mathbb{A}_P, \mathbb{A}_C, J(\mathbb{A}_P), J(\mathbb{A}_C), (A_{s,k}, A_{s,j})_{s \in S}, C_k\right]_{k \in K,\, \mathbb{A}_P \subset \mathcal{A}_P,\, \mathbb{A}_C \subset \mathcal{A}_C,\, j \in J}$.

3. Equilibrium

In order to define equilibrium, we must make clear what the agents optimality condition is in the context of our general preferences. Although the preferences depend only on consumption bundles and real default, the levels of default are not direct decision variables of the agents. However, since each agent is rational, he can perfectly foresee the real default generated by any allocation in his budget set, allowing him to realize which allocations are strictly preferred to any consumption-financial allocation.

More formally, given commodity prices for the states of nature in the second period, $p_{-0} = [(p_s)_{s \in S}]$, when an agent h chooses a short position φ_k^h on primitive k and chooses payments $(\delta_{s,k}^h)_{s \in S}$, he knows the total *real* amount of his default, at state $s \in S$, is given by

$$D_{s,k}(p_s, \varphi_k^h, \delta_{s,k}^h) = \frac{\left[p_s A_{s,k} \varphi_k^h - \delta_{s,k}^h\right]^+}{p_s\, v_s},$$

where, as in Dubey, Geanakoplos and Shubik [7], the vector $v_s \in \mathbb{R}_{++}^L$ is exogenously given and allows the agents to measure the amount of default in real terms.

Therefore, as traders are price takers, given a consumption and financial allocation $(x^h, \varphi^h, \delta^h, \theta^h)$ in $\mathbb{X} := X^h \times \mathbb{R}^k \times \mathbb{R}_+^{K \times S} \times \mathbb{R}_+^J$, the agent h can determine the consumption and default allocation (x^h, d^h) generated by $(x^h, \varphi^h, \delta^h, \theta^h)$, which is given by

$$T\left(p_{-0}, x^h, \varphi^h, \delta^h, \theta^h\right) := \left(x^h, D_{s,k}(p_s, \varphi_k^h, \delta_{s,k}^h)\right)_{(s,k) \in S \times K}.$$

Thus, each agent knows the set of feasible consumption and default allocations $T(p_{-0},\, B^h(p, q, r))$. Therefore, given prices (p, q) and anonymous rates of payment r, an allocation $(x^h, \varphi^h, \delta^h, \theta^h)$ will be optimal for the agent h if and only if $\left[Q^h \circ T(p_{-0}, x^h, \varphi^h, \delta^h, \theta^h)\right] \cap \left[T(p_{-0},\, B^h(p, q, r))\right] = \emptyset$. That is, it does not exist an allocation in the budget set that generates a consumption-default allocation that is strictly preferred to the consumption and default allocation generated by the allocation

$(x^h, \varphi^h, \delta^h, \theta^h)$.[5]

We can now define the concept of equilibrium in the economy with asset-backed securitization.

DEFINITION 1. *An equilibrium for the economy* $\mathcal{E}(S^*, \mathcal{H}, \mathcal{L}, \mathcal{F})$ *is given by prices and rates of payment* $[\overline{p}, \overline{q}, \overline{r}] \in \mathbb{P} := \mathbb{R}_+^{L\times S*} \times \mathbb{R}_+^J \times \mathbb{R}_+^K \times [0,1]^{S\times J}$, *and allocations* $[\overline{x}^h, \overline{\varphi}^h, \overline{\delta}^h, \overline{\theta}^h] \in \mathbb{X}$, *for each agent* $h \in H$, *such that*

(A) *For each agent* $h \in H$, $(\overline{x}^h, \overline{\varphi}^h, \overline{\delta}^h, \overline{\theta}^h) \in B^h(\overline{p}, \overline{q}, \overline{r})$.

(B) *Physical Markets are cleared,*

$$\sum_{h\in H} \overline{x}_0^h = W_0, \qquad \sum_{h\in H} \overline{x}_s^h = W_s + Y_s(W_0), \quad \forall s \in S.$$

(C) *Agents make optimal choices,*

$$\left[Q^h \circ T(\overline{p}_{-0}, \overline{x}^h, \overline{\varphi}^h, \overline{\delta}^h, \overline{\theta}^h)\right] \cap \left[T\left(\overline{p}_{-0},\, B^h(\overline{p}, \overline{q}, \overline{r})\right)\right] = \emptyset \quad \forall h \in H.$$

(D) *For each* $\mathbb{A}_i \subset \mathcal{A}_i$, *with* $i \in \{P, C\}$, *the value of derivatives aggregate purchases must match the value of the aggregate short sales,*

$$\sum_{h\in H} \sum_{j\in J(\mathbb{A}_i)} \overline{q}_j\, \overline{\theta}_j^h = \sum_{h\in H} \sum_{k\in\mathbb{A}_i} \overline{q}_k \overline{\varphi}_k^h,$$

(E) *At each state* $s \in S$ *and for each class* $\mathbb{A}_i \subset \mathcal{A}_i$, *with* $i \in \{P, C\}$, *the total payments of the derivatives must be equal to the total deliveries made by the borrowers,*

$$\sum_{h\in H} \sum_{j\in J(\mathbb{A}_i)} \overline{r}_{s,j}\, \overline{p}_s A_{s,j} \overline{\theta}_j^h = \sum_{h\in H} \sum_{k\in\mathbb{A}_i} \overline{\delta}_{s,k}^h.$$

(F) *At each state* $s \in S$ *and for each class* $\mathbb{A}_i \subset \mathcal{A}_i$, *with* $i \in \{P, C\}$, *the payment rates must be consistent with the financial structure,*

$$\overline{r}_{s,j} = \overline{r}_{s,\mathbb{A}_P}, \quad \forall j \in J(\mathbb{A}_P),\ \forall\, \mathbb{A}_P \subset \mathcal{A}_P,$$

$$(\overline{r}_{s,j})_{j\in J(\mathbb{A}_C)} \in \mathcal{R}(\mathbb{A}_C), \quad \forall\, \mathbb{A}_C \subset \mathcal{A}_C.$$

REMARK 3 (Equilibrium Rates of Payment and Assets Pricing). It follows from equilibrium conditions (E) and (F) that, if a pass-through derivative $j \in J(\mathbb{A}_P)$ is negotiated and, at state $s \in S$, the value of its promises is strictly positive, then the rate of payment $\overline{r}_{s,\mathbb{A}_P}$ is given by the ratio between the total deliveries made by borrowers and the total payments received by lenders. Furthermore,

[5]We have to state our optimality condition as $\left[Q^h \circ T(p_{-0}, x^h, \varphi^h, \delta^h, \theta^h)\right] \cap \left[T(p_{-0},\, B^h(p,q,r))\right] = \emptyset$, since we want to make clear the relationship between our framework and Dubey, Geanakoplos e Shubik [7] model, letting the agents have the same decision variables as in their model. We are aware that some readers could find more natural to allow agents to choose directly the variables (x, φ, d, θ), instead of $(x, \varphi, \delta, \theta)$, where d denotes the real amount of default. In this context, we would have to redefine our budget set, and the optimality condition would be $Q^h(x^h, d^h) \cap B_{x,d}^h(p,q,r) = \emptyset$, where $B_{x,d}^h(p,q,r)$ is the projection of the new budget set in the variables (x, d). However, this approach may generate technical problems on the proof of lower hemicontinuity of budget correspondences, if preferences are defined, as it is natural, only over non-negative amounts of default.

given a tranche $j^m(\mathbb{A}_C)$, the equilibrium rate of payment $\overline{r}_{s,j^m(\mathbb{A}_C)}$, when it is negotiated (and the values of its promises is strictly positive), takes into account the payments made to the previous tranches, in the sense that

$$\overline{r}_{s,j^m(\mathbb{A}_C)} = \max\left\{0\,;\min\left\{\frac{\sum_{h\in H}\sum_{k\in\mathbb{A}_C}\overline{\delta}^h_k - \sum_{i=1}^{m-1}\overline{p}_s\,A_{s,j^i(\mathbb{A}_C)}\sum_{h\in H}\overline{\theta}^h_{j^i(\mathbb{A}_C)}}{\overline{p}_s\,A_{s,j^m(\mathbb{A}_C)}\sum_{h\in H}\overline{\theta}^h_{j^m(\mathbb{A}_C)}};\,1\right\}\right\}.$$

Finally, it is important to remark that the existence of extra-economic enforcements can allow borrowers to raise more capital than the collateral value. However, if extra-economic penalties do not exist, the value of the unitary physical collateral will be, at equilibrium, strictly greater than the value of the asset. In other case, when an agent makes the joint operation of buying the collateral and selling the promises, he has an arbitrage opportunity, since he raises non-negative transfers today, receives non-negative returns tomorrow and has the right to consume the collateral requirements.

4. Collateral Avoids Over-pessimistic Beliefs

Our definition of equilibrium could generate misleading results. When agents are allowed to have pessimistic beliefs about the future derivatives rates of payment, it is always possible to trivially guarantee the existence of an equilibrium. In fact, suppose that the price of primitives and the rates of payment of derivatives are equal to zero, i.e. $(\overline{q}_k, \overline{r}_{s,j})_{(s,j,k)\in S\times J\times K} = 0$. Since an agent h does not expect to receive any payment if he buys a derivative, he has no incentive to do it, so the allocation $\overline{\theta}^h = 0$ is optimal. Similarly, since primitive assets have zero price, $\overline{\varphi}^h = 0$ is optimal for each agent $h \in H$. Furthermore, as agents will not have any promise to pay at the second period, $\overline{\delta}^h_{s,k} = 0$, for all $s \in S$ and $k \in K$, is also optimal. Therefore, the model becomes equivalent to a general equilibrium model with durable goods and without financial markets. Existence of a *pure spot market equilibrium* in this framework is not difficult to prove.

Note that, when over-pessimistic beliefs are allowed, the proof mentioned above would be as good as any other. Thus, it would not be satisfactory to guarantee the existence of equilibrium without excluding this possibility.

It is worth to note that this problem is not idiosyncratic to our model. In fact, it should be considered at every model in which agents take the payment rates of assets promises as given. Although the expected rates of payment are determined endogenously in equilibrium, if derivatives are not traded, any rate of payment is consistent with equilibrium. Thus, agents could be extremely pessimistic, believing that no deliveries would be made in any state, for any asset, which in turn leads to non-negotiation of derivatives.

In their seminal paper, Dubey, Geanakoplos and Shubik [7] address this topic proposing a refined equilibrium concept in order to avoid these over-pessimistic beliefs. They define a ε-boosted equilibrium as an equilibrium of an abstract economy, in which exists an external agent who buys and sells ε units of each asset (that may be interpreted as a government that guarantees an infinitesimal minimum delivery rate), and always delivers the total promises, injecting new commodities in the economy. Therefore, lenders are not over-pessimistic and the rates of payment at each ε-boosted equilibrium are strictly positive. When ε goes to zero, they obtain a refined equilibrium.

In their refinement, Dubey, Geanakoplos and Shubik [7] use the touch of optimism introduced by the ε-agent to banish extremely pessimistic beliefs about the future rates of default. In our model, however, physical collateral requirements introduce a new dimension: it is natural to suppose lenders will expect to receive positive payments when the depreciated collateral bundles of the underling primitives are different from zero. In this sense, collateral avoids over-pessimistic belief without having to use an external agent.

More formally, we propose another refinement concept in which we guarantee that, at each state of nature: (i) equilibrium payment rates are strictly positive for the Pass-through securities that are backed by debt contracts that, independent of the structure of extra-economic penalties, give positive returns, (ii) when primitives associated with a CLO give positive returns, independent of extra-economic enforcements, the most senior tranche, which made non zero promises at this state, has non-zero anonymous payment rate, (iii) when some derivative has a positive rate of payment, at least one of the primitives that backs it has positive price. That is,

Definition 2. *An* equilibrium $[(\overline{p},\overline{q},\overline{r});(\overline{x}^h,\overline{\varphi}^h,\overline{\delta}^h,\overline{\theta}^h)_{h\in H}]$ *is* non-trivial *if the expected payment rates are not over-pessimistic. That is,*

i. *At each state $s\in S$, and for each class $\mathbb{A}_P\subset\mathcal{A}_P$,*

$$\left[\min_{k\in\mathbb{A}_P}\{\overline{p}_sA_{s,k};\overline{p}_sY_sC_k(\overline{p}_0,\overline{q}_k)\}>0\right]\Rightarrow\left[\overline{r}_{s,\mathbb{A}_P}>0\right]\wedge\left[\exists\, k'\in\mathbb{A}_P,\,\overline{q}_{k'}>0\right];$$

ii. *At each state $s\in S$, and for each class $\mathbb{A}_C\subset\mathcal{A}_C$,*

$$\left[\min_{k\in\mathbb{A}_C}\{\overline{p}_sA_{s,k};\overline{p}_sY_sC_k(\overline{p}_0,\overline{q}_k)\}>0\right]\Rightarrow\left[\overline{r}_{s,j^m_{\mathbb{A}_C}}>0,\ \forall m\leq m^\star\right]\wedge\left[\exists\, k'\in\mathbb{A}_C,\,\overline{q}_{k'}>0\right],$$

where $m^\star:=\min\{m\,:\,\|A_{s,j^m(\mathbb{A}_C)}\|_1\neq0\}$.

We are only interested in equilibria in which agents anticipate that derivatives deliveries are strictly positive when physical collateral requirements of underling primitives do not disappear from the economy. Therefore, derivatives have non-zero equilibrium rates of payments whenever the minimum possible payment of their underling primitive assets is strictly positive. Moreover, a class of primitives that backs derivatives with non-trivial rates of payments has at least one asset with non-zero price.

Note that it would not be reasonable to ask agents to expect more optimistic rates of payment, since they do not know what is the total amount of primitives that was sold by the borrowers. In fact, rates of payment depend, in equilibrium, on both the total units of primitives sold and the total units of derivatives bought (see Remark 3 above).

Furthermore, in the framework of Dubey, Geanakoplos and Shubik [7] it is not possible to implement our refinement concept of equilibrium, because their loans are not backed by physical requirements and, hence, the minimum deliveries are always zero.

Finally, note that even with our refinement concept, it is possible that at equilibrium it does not exists a class of primitives that satisfies the conditions stated in Definition 2. In this case, a pure spot market equilibrium can be assured in a trivial manner and, as we say above, our proof is superfluous. Hence, we discuss in Remark 4 (after the statement of the assumptions) the characteristics over the financial structure that guarantee that a family of derivatives has, in any equilibrium, non-trivial rates of payments.

5. Equilibrium Existence

In order to guarantee the existence of a *non-trivial* equilibrium, we will make the following assumptions.

Assumption 1.

For each agent $h \in H$ and each state of nature $s \in S^*$, the income function $m_s^h(p)$ is continuous, strictly positive ($m_s^h(p_s) > 0$) whenever $p_s \neq 0$ and satisfies

$$\sum_{h\in H} m_s^h(p_s) = p_s W_s, \quad \textit{for all} \quad p_s \in \mathbb{R}_+^L.$$

As it was mentioned before, this first hypothesis is weaker than the usual strong survival assumption used in the general equilibrium literature. We are only assuming that whenever the prices are not all equal to zero, agents will have some income. Hence, we are not interested how agents obtain their incomes (we can even think that there exists some kind of central planner that does not allow anyone to starve). Moreover, we also assume that the aggregate income of agents matches the total value of the endowment available in the economy.

Assumption 2.

For each agent $h \in H$, the correspondence $Q^h : X \times \mathbb{R}_+^{K\times S} \twoheadrightarrow X \times \mathbb{R}_+^{K\times S}$ has open graph, is irreflexive (that is, $(x,d) \notin Q^h(x,d)$ for all (x,d)), has convex-values, and satisfies the following conditions:

(i) *strictly monotonicity on $(x_s)_{s\in S^*}$, that is, $(x',d) \in Q^h(x,d)$, for all $x' > x$;*
(ii) *if $(x',d') \in Q^h(x,d)$, then for all $d'' < d'$, (x',d'') also belongs to $Q^h(x,d)$;*

where, given vectors w and z in an Euclidean space, $w > z$ is defined as $w_i \geq z_i$ for all i and $w_{i'} > z_{i'}$ for at least one i'.

As noted above, we are not assuming that individual preferences are complete, transitive nor continuous. Moreover, the assumptions made over correspondences Q^h can appear to be too demanding, since we assume open graph instead of lower hemicontinuity with open values as usual. However, we need this assumption, as well as item (ii), in order to use the Gale and Mas-Collel Fixed Point Theorem in the proof of our main result (see Appendix).

Assumption 3.

The collateral requirements functions $C_k : \mathbb{R}_+^L \times \mathbb{R}_+ \to \mathbb{R}_+^L$ are continuous and different from zero in their domain. Moreover, for each primitive $k \in K$, the function C_k, if it is not constant, satisfies $Y_s C_k(p_0,q_k) \neq 0$ for all $(p_0,q_k) \in \mathbb{R}_+^L \times \mathbb{R}_+$ and for each $s \in S$.

Thus, when collateral requirements depend on price levels, the commodities that are used as guarantees will be non-perishable. Moreover, it is important to remark that physical collateral requirements, which act as enforcement mechanisms in case of default, also guarantee that at equilibrium short sales of primitives are bounded.[6]

ASSUMPTION 4.

Given an agent $h \in H$ and given $\epsilon > 0$, there exists, for each $(x^h, d^h) \in X \times \mathbb{R}_+^{S \times K}$ and for each pair $(s,l) \in S^* \times L$, a constant $Z_{s,l}^h(x^h, d^h, \epsilon) \in \mathbb{R}_{++}$ such that the allocation $(y^h, 0)$, with

$$y_{s',l'}^h = \begin{cases} \epsilon, & \text{if } (s',l') \neq (s,l), \\ Z_{s,l}^h(x^h, d^h, \epsilon), & \text{if } (s',l') = (s,l), \end{cases}$$

is strictly preferred to (x^h, d^h), i.e. $(y^h, 0) \in Q^h(x^h, d^h)$. Moreover, the functions

$$Z_{s,l}^h : X \times \mathbb{R}_+^{K \times S} \times \mathbb{R}_{++} \to \mathbb{R}_{++},$$

are non-decreasing in x, and non-increasing in d.

This assumption guarantees that all equilibrium commodity prices are uniformly bounded from below at each state of nature $s \in S^*$ (see Appendix). This property on prices is sufficient to assure that there exists an equilibrium with non-trivial rates of payment.[7]

ASSUMPTION 5.

Assets are non-trivial, in the sense that, for each $k \in K$ (respectively, for each $j \in J$), the vector of real promises $A_k = (A_{s,k})_{s \in S}$ (respectively, $A_j = (A_{s,j})_{s \in S}$) is different from zero. Moreover, for

[6] If we had let the collateral bundle to be zero (i.e. in case of default, only extra-economic penalties burden agents payments), then we would have to suppose that primitives short sales are exogenously bounded in order to guarantee existence of an equilibrium. Moreover, for this kind of asset, it would never be possible to guarantee, using our proof of equilibrium, that rates of payments of its associated derivatives are strictly positive.

[7] In the framework of Dubey, Geanakoplos and Shubik [7], in which

$$Q^h(x,d) := \left\{ (x', d') : U^h(x') - \sum_{s \in S} \sum_{k \in K} \lambda_{s,k}^h d'_{s,k} > U^h(x) - \sum_{s \in S} \sum_{k \in K} \lambda_{s,k}^h d_{s,k} \right\},$$

where $U^h : \mathbb{R}_+^{L \times S^*} \to \mathbb{R}_+$ and $\lambda_{s,k}^h \in \mathbb{R}_+$, Assumption 4 is implied by their hypothesis in utilities, $\lim_{||z||_\infty \to +\infty} U^h(z) = +\infty$. In fact, our condition is weaker, because we do not need to assume, for instance, that $U^h(a, 0, \dots, 0)$ goes to infinity when $a \in \mathbb{R}_+$ goes to infinity.

Moreover, these authors use the assumption of unbounded utilities to prove that there exists an equilibrium even without interior individual endowments. In our context, Assumption 4 will guarantee that all equilibrium commodity prices are uniformly bounded from below and, therefore, there exists $\underline{p} > 0$ such that, given equilibrium commodity prices $\overline{p}$, we have $\overline{p}_{s,l} \geq \underline{p}$, for all states of nature $s \in S^*$ and for all commodity $l \in L$ (see Lemma 4 in the Appendix). Thus, fixing endowments $w^h = (w_s^h)_{s \in S^*} \in \mathbb{R}_+^{L \times S^*}$, that satisfy for each state of nature $\sum_{h \in H} w_s^h \gg 0$, and restating our Assumption 1, in order to require that the last condition $\sum_{h \in H} m^h(p_s) = p_s W_s$ only holds for the prices $p_s \geq \underline{p}$. We can define income functions as $m_s^h(p_s) = \sum_{l \in L} \max\{p_{s,l}, \underline{p}\} w_{s,l}^h$ in order to guarantee, as consequence of our main result, that there exists a non-trivial equilibrium for an economy in which agents are endowed by physical bundles that do not need to be interior points of $\mathbb{R}_+^L$. Note that, in our context in which commodities may be durable, this property is particularly interesting, because it allows some agents to be endowed, at the states of nature in the second period, only with perishable commodities.

each class of primitives $\mathbb{A}_i \subset \mathcal{A}_i$, with $i \in \{P,C\}$, we have that

$$\left[\sum_{k\in\mathbb{A}_i}(A_{s,k})_l \neq 0\right] \Leftrightarrow \left[\sum_{j\in J(\mathbb{A}_i)}(A_{s,j})_l \neq 0\right], \ \forall s \in S, \forall\, l \in L.$$

This last assumption guarantees that, independently of the price level, one derivative has positive real promises if and only if at least one primitive also has it.

We can now state our main result,

Theorem. *Under Assumptions 1-5 our economy $\mathcal{E}(S^*,\mathcal{H},\mathcal{L},\mathcal{F})$ has a non-trivial equilibrium.*

Sketch of the proof: We guarantee, in the Appendix, the existence of a non-trivial equilibrium following the steps below:

⋄ We define, for each agent $h \in H$, a correspondence that, given a price vector, associates, to each consumption-financial allocation, the set of consumption-financial allocations that are strictly preferred to it. These correspondences are lower hemicontinuous, strictly monotonic, irreflexive and have open and convex values (see Proposition 1). Moreover, in order to guarantee that agents budget set correspondences are lower hemicontinuous, we suppose that (i) first period commodities and derivatives prices are in the simplex, (ii) primitive assets prices satisfy $\sum_{k\in\mathbb{A}_i} q_k \geq \sum_{j\in J(\mathbb{A}_i)} q_j$ for any class $\mathbb{A}_i \subset \mathcal{A}_i$, and (iii) the commodities prices at each state in the second period belong to the simplex.

⋄ As we work in an economy in which agents are characterized by non-ordered preferences, we find an equilibrium using Gale-Mas-Colell Fixed Point Theorem. Thus, we need to truncate our economy. In this direction, we prove in Lemma 1 that equilibrium allocations are bounded. Hence we define, for each $M = (M_1,M_2) \in \mathbb{R}^2_+$ with $M_2 > M_1$, a truncated economy $\mathcal{E}_M$ that coincides with the original economy, but has the endogenous variables truncated in the following way: (i) short sales of primitives, purchases of derivatives and deliveries are bounded using upper bounds found on Lemma 1; (ii) consumption allocations are bounded by M_1; and (iii) the rates of payment of each derivative are restricted to belong to $[\beta,1]$, where the lower bound β is equal to $\frac{1}{M_1}$, if it satisfies one of the refinement conditions, and equal to $\frac{1}{M_2}$ in other case. Note that payment rates of CLO, associated to $\mathbb{A}_C$, are not restricted to belong to $\mathcal{R}(\mathbb{A}_C)$, because this set is not convex.

⋄ Furthermore, for each abstract economy, we define reaction correspondences for the agents, and abstract reaction correspondences for auctioneers, who choose prices (for commodities and assets) and rates of payment for the derivatives. The same auctioneer fixes the rates of payment for a family of Pass-troughs and different auctioneers fix the rate of payment of each tranche in the same CLO. Using Gale-Mas-Colell Fixed Point Theorem, we prove that there exists, for each vector (M_1,M_2) an *equilibrium for the truncated economy* $\mathcal{E}_M$, in the sense that all reaction correspondences have empty-value (see Lemma 2).

⋄ Now, for M_1 sufficiently large, we prove that: (i) agents allocations are feasible and optimal in the truncated budget set (Lemma 3); (ii) market clearing conditions in the first period are satisfied for physical and financial markets (Lemma 3); and (iii) commodities prices are uniformly bounded from below at each state of nature $s \in S^*$ (Lemma 4).

$\diamond$ Moreover, Lemma 5 and Lemma 6 prove, using the reaction correspondences of the auctioneers, for M_1 sufficiently large, that (i) Pass-through rates of payment satisfy equilibrium condition (F); (ii) CLO payment rates belong to a truncated space of payments (that converges to the space $\mathcal{R}(\mathbb{A}_C)$ when M_2 goes to infinity); and (iii) the excess of payments received by the lenders over the deliveries made by the borrowers is bounded by a constant multiplied by $\frac{1}{M_2}$.

$\diamond$ In Lemma 7 we prove that the physical markets excess of demand, at each state in the second period, is also bounded by a constant multiplied by $\frac{1}{M_2}$. Furthermore, we guarantee that agents allocations, obtained as equilibrium for the truncated economies $\mathcal{E}_M$, are optimal choices in the original economy for each M.

$\diamond$ Finally, as we have two degrees of freedom, M_1 and M_2, we prove in Lemma 8 that, for a sufficiently large M_1, when M_2 goes to infinity the equilibrium of the truncated economies converges to an equilibrium of the original economy. As payment rates of derivatives associated with primitives that satisfy our refinement condition are bounded from below by $\frac{1}{M_1}$, and optimality condition implies that there exists at least one primitive that has positive price, we guarantee that this equilibrium is non-trivial. $\square$

Particularly, when the seizure of the collateral requirements, which are independent of prices, is the only enforcement in case of default, our main result guarantees that there exists a non-trivial equilibrium in the context of the original Geanakoplos e Zame [11] model, allowing asset-backed securitization and non-ordered preferences. Moreover, when there is only one derivative associated with each primitive, we obtain an extension of Geanakoplos and Zame [11] to allow non-ordered preferences.[8]

REMARK 4 (Assets with Non-trivial Rates of Payment). We can guarantee, independently of the equilibrium prices, that some derivatives always have positive rates of payment, even if they are not negotiated. In fact, as we suppose that preferences are strictly monotonic on consumption, equilibrium commodity prices (if they exist) will be strictly positive, $\overline{p} \gg 0$, which implies that for each class of primitives $\mathbb{A}_i$, with $i \in \{P, C\}$, a necessary and sufficient condition to guarantee that

$$\min\{\overline{p}_s A_{s,k}; \overline{p}_s Y_s C_k(\overline{p}_0, \overline{q}_k)\} > 0, \;\; \forall k \in \mathbb{A}_i \,,$$

is that $\min_{k\in\mathbb{A}_i}\{||A_{s,k}||_1; ||Y_s C_k(\overline{p}_0, \overline{q}_k)||_1\} > 0$. Now, it follows from Assumption 3 that this last condition does not depend on $(\overline{p}_0, \overline{q}_k)$. In fact, if collateral requirements are price dependent, we know that their depreciated bundle is, at each state of nature, different from zero, which implies that the condition above is equivalent to $\min_{k\in\mathbb{A}_i}\{||A_{s,k}||_1\} > 0$. Hence, when collateral are fixed, as in Geanakoplos and Zame [11], a family of derivatives will have positive rates of payment at

[8] In the context of endogenous collateral models, Martins-da-Rocha and Torres-Martínez [13] also obtain this last result as a particular case of their main theorem.

some state $s \in S$ if both $\min_{k \in \mathbb{A}_i} \{||A_{s,k}||_1\} > 0$ and $\min_{k \in K} ||Y_s C_k||_1 \neq 0$. Therefore, under our assumptions, the requirement that guarantees that a family of derivatives has a non-zero vector of rates of payment is independent of equilibrium prices.

6. Concluding Remarks

In this paper, we prove that the financial structures of asset-backed securitization are consistent with a general equilibrium model in which assets are subject to default and borrowers are enforced by the seizure of collateral requirements or by extra-economic penalties. Equilibrium exists even when agents are only characterized by non-ordered preferences, which internalize extra-economic punishments. Furthermore, we propose a refinement concept of equilibrium in order to avoid the absence of asset trading as consequence of over-pessimism about rates of default. Thus, physical collateral requirements have an important role in order to avoid unduly pessimistic belief on derivatives rates of default.

Our framework may be extended in some directions. A natural one would be to consider a multi-period model with long-lived assets. In this context, if we permit borrowers to prepay their debts before the contracted period, two interesting points would appear. First, how these prepayments are made and that incentives the borrowers have to make them? Second, how the financial intermediaries would reinvest the amount prepaid? Thus, it would be necessary to model the decision making of the issuers, endogenizing the asset structure, as in DeMarzo and Duffie [5], Allen and Gale [1], and Diamond [6].

Although in our model collateral requirements can depend on prices, agents do not have the possibility to choose personalized guarantees. Thus, it is interesting to allow borrowers to endogenously choose this guarantees in a context with asset-backed securitization. In the particular case in which primitives within the same class differ from each other only on physical collateral requirements, we can argue as in Geanakoplos [10] that this structure approximates an endogenous collateral context. In fact, assets will be priced in equilibrium and agents will choose the pair of primitive price and collateral bundle that best suits their interests. Of course, a more realistic endogenous collateral model, though, would have to allow agents to directly choose the physical collateral requirements, as in Araujo, Fajardo and Páscoa [2] and Martins-da-Rocha and Torres-Martínez [13].

Furthermore, in DeMarzo and Duffie [5], Araujo, Fajardo and Páscoa [2], and Martins-da-Rocha and Torres-Martínez [13] the promises made by the borrowers are pooled in only one nominal security, which incorporates in their promises the default given by the original claims. In this context, these models suppose that financial intermediaries issue *endogenous asset-backed derivatives*. We can allow this type of structure and the existence of equilibrium will be a direct consequence of our main result, after redefinition of some variables. In fact, when classes of primitives are pooled in only one derivative, it is sufficient to suppose that agents, instead of taking anonymous rates of payment as given, observe nominal promises made by the derivatives.

Thus, our model of Asset-Backed Securitization provides a natural framework to study these extensions and to analyze the advantages of asset backed securitization in order to promote efficiency.

APPENDIX: PROOF OF THE THEOREM

In order to guarantee the equilibrium existence, we restrict our space of prices. In the first period, we consider prices (p_0, q_K, q_J) that belong to the convex-compact set,

$$\Xi = \left\{ (p_0, q_K, q_J) \in \mathbb{R}_+^L \times [0,1]^K \times \mathbb{R}_+^J : (p_0, q_J) \in \Delta_+^{\#L+\#J-1}, \sum_{j \in J(\mathbb{A}_i)} q_j \leq \sum_{k \in \mathbb{A}_i} q_k, \;\; \forall \mathbb{A}_i \subset \mathcal{A}_i \right\},$$

where $q_K = (q_k)_{k \in K}$, $q_J = (q_j)_{j \in J}$ and, as usual, $\mathbb{A}_i$ denotes a generic class of primitives of $\mathcal{A}_i$, with $i \in \{P, C\}$. Moreover, we restrict the commodity prices p_s, at each state of nature in the second period, to belong to $\Delta_+^{\#L-1}$. For convenience of notation, we denote a generic vector of commodity-financial prices and derivatives rates of payment by $\pi = (p, q, r)$, a generic individual allocation for an agent h by $\eta^h = (x^h, \varphi^h, \delta^h, \theta^h)$, and by $\eta = (\eta^h)_{h \in H}$ a generic vector of allocations in the economy. Also, let Ξ_k be the projection of the set Ξ into the commodity prices and the price of the primitive k, that is,

$$\Xi_k = \left\{ (p_0, q_k) \in \mathbb{R}_+^L \times [0,1] : \exists \, (q_J, q_{k'})_{k' \neq k}, (p_0, q_K, q_J) \in \Xi \right\}.$$

Moreover, without loss of generality, we suppose that $\sum_{k \in \mathbb{A}_i} A_{s,k} \leq \sum_{j \in J(\mathbb{A}_i)} A_{s,j}$ for each $\mathbb{A}_i \subset \mathcal{A}_i$ with $i \in \{P, C\}$ and for each $s \in S$.

REMARK 5. Note that, if we prove that there always exists a non-trivial equilibrium $(\overline{\pi}, \overline{\eta})$ for any economy $\mathcal{E}$ that satisfies $\sum_{k \in \mathbb{A}_i} A_{s,k} \leq \sum_{j \in J(\mathbb{A}_i)} A_{s,j}$, for each $\mathbb{A}_i \subset \mathcal{A}_i$ with $i \in \{P, C\}$ and for each $s \in S$, it is always possible to find a non-trivial equilibrium for any economy $\mathcal{E}'$ in which primitive and derivative promises only satisfy Assumption 5. In fact, for such $\mathcal{E}'$ we have that

$$\left[\sum_{k \in \mathbb{A}_i} (A'_{s,k})_l \neq 0 \right] \Leftrightarrow \left[\sum_{j \in J(\mathbb{A}_i)} (A'_{s,j})_l \neq 0 \right], \;\; \forall s \in S, \forall \, l \in L,$$

and consequently there exists $\lambda \in \mathbb{R}_{++}$ such that $\sum_{k \in \mathbb{A}_i} A'_{s,k} \leq \sum_{j \in J(\mathbb{A}_i)} \lambda A'_{s,j}$, for each $\mathbb{A}_i \subset \mathcal{A}_i$ with $i \in \{P, C\}$ and for each $s \in S$. If there is an equilibrium $(\overline{\pi}, \overline{\eta})$ for an economy $\mathcal{E}$, which is equal to $\mathcal{E}'$ except for the derivatives promises, defined as $A_{s,j} = \lambda A'_{s,j}$, we can consider the allocation $(\overline{\pi}', \overline{\eta}')$ given by $(\overline{p}', \overline{q}'_K, \overline{r}'; \overline{x}', \overline{\varphi}', \overline{\delta}') = (\overline{p}, \overline{q}_K, \overline{r}; \overline{x}, \overline{\varphi}, \overline{\delta})$, $\overline{\theta}' = \lambda \, \overline{\theta}$ and $\overline{q}'_J = \frac{1}{\lambda} \, \overline{q}_J$. One can easily verify that the allocation $(\overline{\pi}', \overline{\eta}')$ is an equilibrium for the economy $\mathcal{E}'$.

We define the *decision correspondence* $\Psi^h : (\Delta_+^{\#L-1})^S \times \mathbb{X} \twoheadrightarrow \mathbb{X}$, by

$$\Psi^h(p_{-0}, \eta^h) := (T_{p_{-0}})^{-1} \circ Q^h \circ T(p_{-0}, \eta^h),$$

where the continuous function $T_{p_{-0}} : \mathbb{X} \to X \times \mathbb{R}_+^{K \times S}$ is defined by $T_{p-0}(\eta^h) := T(p_{-0}, \eta^h)$. It follows that, *given prices (p, q) and rates of payment r, an allocation η^h is optimal for the agent $h \in H$ if and only if $\Psi^h(p_{-0}, \eta^h) \cap B^h(p, q, r) = \emptyset$.*

PROPOSITION 1. *Under Assumption 2, for each agent $h \in H$, the correspondence Ψ^h is lower hemicontinuous, has convex and open values, is strictly monotonic on $(x_s)_{s \in S^*}$, and is irreflexive, in the sense that $\eta^h \notin \Psi^h(p_{-0}, \eta^h)$, for all $(p_{-0}, \eta^h) \in (\Delta_+^{\#L-1})^S \times \mathbb{X}$.*

PROOF: For convenience of notation, define $\mathbb{F} := (\Delta_+^{\#L-1})^S$. Also, we denote by f a generic element of $\mathbb{F}$. Using our notation, $\Psi^h(f, \eta^h) = (T_f)^{-1} \circ Q^h \circ T(f, \eta^h)$, *Step 1 :* Ψ^h *has open values.* Fix a vector $(f,\eta^h)\in\mathbb{F}\times\mathbb{X}$. Since Q^h has open values, $Q^h(T(f,\eta^h))$ is an open set. As T_f is continuous, $(T_f)^{-1}(Q^h(T(f,\eta^h)))$ is also open, which implies that Ψ^h has open values.

Step 2 : Ψ^h *has convex values.* Fix a vector $(f,\eta^h)\in\mathbb{F}\times\mathbb{X}$. It is sufficient to show that, given $\lambda\in[0,1]$ and η_1^h and η_2^h in $\Psi^h(f,\eta^h)$, we have that $\left[\lambda\eta_1^h+(1-\lambda)\eta_2^h\right]\in(T_f)^{-1}\left(Q^h(T(f,\eta^h))\right)$. Since Q^h is convex-valued, we know that $\lambda T(f,\eta_1^h)+(1-\lambda)T(f,\eta_2^h)\in Q^h(T(f,\eta^h))$. Moreover, by definition of the function T, there exists $(x_1,x_2)\in X\times X$ such that

$$T(f,\eta_i^h)=\left(x_i,\left(D_{s,k}(f,\eta_i^h)\right)_{(s,k)\in S\times K}\right),\quad i\in\{1,2\}.$$

As the functions $D_{s,k}(f,\cdot)$ are convex, we have that

$$\begin{aligned}
\lambda T(f,\eta_1^h)+(1-\lambda)T(f,\eta_2^h) &= \left(\lambda x_1+(1-\lambda)x_2,\left[\lambda D_{s,k}(f,\eta_1^h)+(1-\lambda)D_{s,k}(f,\eta_2^h)\right]_{(s,k)\in S\times K}\right)\\
&\geq \left(\lambda x_1+(1-\lambda)x_2,\left[D_{s,k}(f,\lambda\eta_1^h+(1-\lambda)\eta_2^h)\right]_{(s,k)\in S\times K}\right)\\
&= T(f,\lambda\eta_1^h+(1-\lambda)\eta_2^h)
\end{aligned}$$

Therefore, it follows from item (ii) in Assumption 2 that $T(f,\lambda\eta_1^h+(1-\lambda)\eta_2^h)$ belongs to $Q^h(T(f,\eta^h))$, which implies that $\left(\lambda\eta_1^h+(1-\lambda)\eta_2^h\right)\in(T_f)^{-1}\left(Q^h(T(f,\eta^h))\right)$.

Step 3 : Ψ^h *is lower-hemicontinuous.* It is sufficient to show that given an (relative) open set $U\subset\mathbb{X}$, the lower inverse $(\Psi^h)^-[U]=\{(f,\eta^h)\in\mathbb{F}\times\mathbb{X}\,:\,\Psi^h(f,\eta^h)\cap U\neq\emptyset\}$, is an (relative) open set of $\mathbb{F}\times\mathbb{X}$. If $(\Psi^h)^-[U]$ is empty, the proof is immediate. Thus, suppose that $(\Psi^h)^-[U]\neq\emptyset$ and fix a vector $(\overline{f},\overline{\eta}^h)\in(\Psi^h)^-[U]$. We are interested in proving that there exists $\overline{\mu}>0$ such that

$$V_{\overline{\mu}}(\overline{f},\overline{\eta}^h)\bigcap(\mathbb{F}\times\mathbb{X})\subset(\Psi^h)^-[U],$$

where $V_\mu(f,\eta^h)$ denotes the open neighborhood (in the Euclidean norm) of (f,η^h) with radius μ. Now, fix a vector $\hat{\eta}^h\in\Psi^h(\overline{f},\overline{\eta}^h)\cap U$.
We know that $\hat{\eta}^h\in U$ and $T(\overline{f},\hat{\eta}^h)\in Q^h\circ T(\overline{f},\overline{\eta}^h)$, which implies that $\left(T(\overline{f},\overline{\eta}^h),T(\overline{f},\hat{\eta}^h)\right)\in\text{Graph }Q^h$. As the correspondence Q^h has open graph, there exists $\nu>0$ such that

$$V_\nu\left(T(\overline{f},\overline{\eta}^h),T(\overline{f},\hat{\eta}^h)\right)\bigcap\left[(X\times\mathbb{R}_+^{K\times S})\times(X\times\mathbb{R}_+^{K\times S})\right]\subset\text{Graph }Q^h.$$

Moreover, as the function T is continuous, there exists $\overline{\mu}>0$ such that, if $(f',\eta'^h)\in V_\mu(\overline{f},\overline{\eta}^h)\cap[\mathbb{F}\times\mathbb{X}]$ then

$$\left(T(f',\eta'^h),T(f',\hat{\eta}^h)\right)\in V_\nu\left(T(\overline{f},\overline{\eta}^h),T(\overline{f},\hat{\eta}^h)\right)\bigcap\left[(X\times\mathbb{R}_+^{K\times S})\times(X\times\mathbb{R}_+^{K\times S})\right].$$

Therefore, for all $(f',\eta'^h)\in V_{\overline{\mu}}(\overline{f},\overline{\eta}^h)\cap[\mathbb{F}\times\mathbb{X}]$, we have that $T(f',\hat{\eta}^h)\in Q^h\left(T(f',\eta'^h)\right)$, which implies $\hat{\eta}^h\in\Psi^h(f',\eta'^h)$. Finally, as $\hat{\eta}^h\in U$, the allocation (f',η'^h) belongs to $(\Psi^h)^-[U]$, which concludes the proof of this step.

The fact that Ψ^h is strictly monotonic and irreflexive follows directly from properties on Q^h. □

It is useful to define, for each agent $h\in H$, a set function $\hat{\Psi}^h:\left(\Delta_+^{\#L-1}\right)^S\times\mathbb{X}\twoheadrightarrow\mathbb{X}$, called *augmented decision correspondence*, as the set function that associates to each vector (p_{-0},η^h) the collection of allocations $\hat{\eta}^h$ that satisfies

$$\hat{\eta}^h=\lambda\tilde{\eta}^h+(1-\lambda)\eta^h\text{ for }0<\lambda\leq1,\ \tilde{\eta}^h\in\Psi^h(p_{-0},\eta^h).$$

Note that $\Psi^h(p_{-0},\eta^h) \subset \hat{\Psi}^h(p_{-0},\eta^h)$. Moreover, as noted in Gale-Mas-Colell [8, 9], the correspondence $\hat{\Psi}^h$ preserves all properties of Ψ^h: it is irreflexive, strictly monotonic and lower hemicontinuous, with open and convex values (see Proposition above).

LEMMA 1. *Given prices and rates of payment $\pi = (p,q,r)$, if individual allocations $(\eta^h)_{h\in H}$ satisfy equilibrium conditions (A)-(C) then, for each agent $h \in H$, the vector $(x^h,\varphi^h,\delta^h) \in X \times \mathbb{R}^K_+ \times \mathbb{R}^{K\times S}_+$ is bounded.*

PROOF: Let $\eta = (x^h,\varphi^h,\delta^h,\theta^h)_{h\in H}$ be a vector that satisfies the equilibrium conditions (A)-(C). Condition (B) implies that, at the first period, $\sum_{h\in H} x^h_0 = W_0$. As each term in the left hand side of the equality above is non-negative, it follows that, for each commodity $l \in L$ and for each agent $h \in H$, the consumption allocation satisfies $x^h_{0,l} \leq W_{0,l}$. Thus, agents consumption bundles are bounded at $t = 0$. Moreover, it follows from Condition (A) that $\eta^h \in B^h(\pi)$ and, therefore,

$$\sum_{k\in K} C_{k,l}(p_0,q_k)\varphi^h_k \leq x^h_{0,l} \leq W_{0,l}. \tag{5}$$

Since we restrict prices (p_0,q_k) to the compact set Ξ_k, it follows from Assumption (3) that there exists a finite lower bound $\underline{c}_k = \min_{(p_0,q_k)\in\Xi_k} \sum_{l\in L} C_{k,l}(p_0,q_k) > 0$. Then, summing over $l \in L$, we have from equation (5) that

$$\varphi^h_k \leq \frac{1}{\underline{c}_k}\sum_{l\in L} W_{0,l} =: \Omega_k, \tag{6}$$

which implies that short sales are bounded. Now, equilibrium condition (B) guarantees that, at each state $s \in S$, $\sum_{h\in H} x^h_s = W_s + Y_s W_0$. Since each term on left hand side, in the last equation, is non negative, it follows that individual consumptions bundles, x^h_s, are bounded.

Finally, as we restrict commodity prices, at each state of nature $s \in S$, to belong to the simplex $\Delta^{\#L-1}_+$, the value of primitive promises, $p_s A_{s,k}\varphi^h_k$, is bounded for each $(h,k) \in H \times K$. Thus, it follows from equilibrium condition (C) that borrowers do not have any incentive, at the optimum, to pay more than the face value of the original promises. Therefore, payments δ^h are bounded from above, node by node, primitive by primitive. □

Now, as consumption allocations, short sales positions and payments are bounded at equilibrium (if it exists), we will truncate endogenous variables in order to find an optimal allocation for the economy.

Our goal is to prove that, given upper and lower bounds on allocations, there exists an equilibrium for a truncated economy (as defined below). Furthermore, we show that this truncated equilibria allocations converge, when the appropriated limit is taken, to an equilibrium allocation of our original economy $\mathcal{E}(S^*,\mathcal{H},\mathcal{L},\mathcal{F})$.

THE TRUNCATED ECONOMY $\mathcal{E}_M$. We define, for each $M \in \mathcal{M} = \{(M_1,M_2) \in \mathbb{R}^2_{++} : M_1 < M_2\}$, a truncated economy $\mathcal{E}_M$ in which the structure of uncertainty and the physical markets are the same as in $\mathcal{E}(S^*,\mathcal{H},\mathcal{L},\mathcal{F})$. Each agent $h \in H$ can demand commodities, can sell primitives $k \in K$ and can buy derivatives $j \in J$ restricted to the space of allocations $\mathbb{X}_M$, which is given by the set of vectors $\eta^h = (x^h,\varphi^h,\delta^h,\theta^h) \in \mathbb{X}$ that satisfies

$$\|x^h\|_\infty \leq M_1, \quad \|\varphi^h\|_\infty \leq 2\Omega, \quad \|\theta^h\|_\infty \leq 2\,(\#H)\,\Omega \quad \text{and} \quad \|\delta^h\|_\infty \leq 2\Omega \max_{(s,k)\in S\times K} \|A_{s,k}\|_1,$$

where $\| \, \|_\infty$ denotes the sup-norm and $\Omega := \max_{k\in K}\Omega_k$ is the maximum of short sales upper bounds defined at equation (6) on Lemma 1. Moreover, in order to guarantee the existence of a non-trivial equilibrium (as defined at Section 4), we need to find a lower bound, above from zero, for the anonymous rates of payment of the derivatives:

- ◇ Given a class $\mathbb{A}_C \subset \mathcal{A}_C$ of primitives, we define the *truncated space of CLO payment rates* as the set of vectors $(r_{s,j^m_{\mathbb{A}_C}}) \in \mathbb{R}^{n(\mathbb{A}_C)}_+$ that belongs to

$$\Upsilon^s_M(\mathbb{A}_C) := \prod_{m=1}^{n(\mathbb{A}_C)} [\beta^{s,m}_M(\mathbb{A}_C), 1]\,,$$

where, denoting by $\overline{c}_{s,k} := \max_{(p_0,q_k)\in\Xi_k} \|Y_s C_k(p_0,q_k)\|_1$,

$$\beta^{s,1}_M(\mathbb{A}_C) = \begin{cases} \frac{1}{M_1}, & \text{if } \min_{k\in\mathbb{A}_C}\{||A_{s,k}||_1, \overline{c}_{s,k}\} > 0; \\ \frac{1}{M_2}, & \text{in other case,} \end{cases}$$

and for $m > 1$,

$$\beta^{s,m}_M(\mathbb{A}_C) = \begin{cases} \frac{1}{M_1}, & \text{if } \left[\min_{k\in\mathbb{A}_C}\{||A_{s,k}||_1, \overline{c}_{s,k}\} > 0\right] \wedge \left[||A_{s,j^{m'}(\mathbb{A}_C)}||_1 = 0,\ \forall m' < m\right]; \\ \frac{1}{M_2}, & \text{in other case.} \end{cases}$$

- ◇ Given a class $\mathbb{A}_P \subset \mathcal{A}_P$ of primitives, we define the *truncated space of Pass-through payment rates* as

$$\Upsilon^s_M(\mathbb{A}_P) = \left\{ r = (r_i) \in [\beta^s_M(\mathbb{A}_C), 1]^{n(\mathbb{A}_P)} \,:\, r_i = r_{i'},\ 1 \leq i, i' \leq n(\mathbb{A}_P) \right\},$$

where, similar to the case of Collateralized Loan Obligations,

$$\beta^s_M(\mathbb{A}_P) = \begin{cases} \frac{1}{M_1}, & \text{if } \min_{k\in\mathbb{A}_P}\{||A_{s,k}||_1, \overline{c}_{s,k}\} > 0; \\ \frac{1}{M_2}, & \text{in other case.} \end{cases}$$

The space of prices and rates of payments $\pi = (p, q, r)$ is given, in our truncated economy $\mathcal{E}_M$, by

$$\mathbb{P}_M := \Xi \times (\Delta^{\#L-1}_+)^S \times \prod_{i\in\{P,C\}} \prod_{\mathbb{A}_i\subset\mathcal{A}_i} \prod_{s\in S} \Upsilon^s_M(\mathbb{A}_i).$$

Moreover, for a given vector of prices and anonymous payment rates $\pi \in \mathbb{P}_M$, let $B^h_M(\pi) = B^h(\pi) \cap \mathbb{X}_M$ be the *truncated* budget set. For each agent h, we define the *truncated augmented decision correspondence* $\hat{\Psi}^{h,M} : (\Delta^{\#L-1}_+)^S \times \mathbb{X}_M \twoheadrightarrow \mathbb{X}_M$ as the restriction of the correspondence $\hat{\Psi}^h$ to the truncated space of allocations $\mathbb{X}_M$.

Now, associated to each agent $h \in H$, we define a reaction correspondence $\psi^h_M : \mathbb{P}_M \times \mathbb{X}^H_M \twoheadrightarrow \mathbb{X}_M$ via

$$\psi^h_M(\pi,\eta) = \begin{cases} \dot{B}^h_M(\pi) & \text{if} \quad \eta^h \notin B^h_M(\pi), \\ \dot{B}^h_M(\pi) \cap \hat{\Psi}^{h,M}(p_{-0},\eta^h) & \text{if} \quad \eta^h \in B^h_M(\pi), \end{cases}$$

where $\dot{B}^h_M(\pi)$ denotes the interior of $B^h_M(\pi)$ relative to $\mathbb{X}_M$. Reaction correspondences are also defined for each state $s \in S^*$. Let $\psi^0_M : \mathbb{P}_M \times \mathbb{X}^H_M \twoheadrightarrow \Xi$ be

$$\psi^0_M(\pi,\eta) = \left\{ (p'_0, q'_K, q'_J) \,:\, p'_0 \left[\sum_{h\in H} x^h_0 - W_0\right] + \sum_{i\in\{P,C\}} \left[\sum_{\mathbb{A}_i\subset\mathcal{A}_i} \left(\sum_{j\in J(\mathbb{A}_i)} q'_j \sum_{h\in H} \theta^h_j - \sum_{k\in\mathbb{A}_i} q'_k \sum_{h\in H} \varphi^h_k \right)\right] > 0 \right\},$$

and, for each $s \in S$, let $\psi^s_M : \mathbb{P}_M \times \mathbb{X}^H_M \twoheadrightarrow \Delta^{\#L-1}_+$ be

$$\psi^s_M(\pi,\eta) = \left\{ p'_s \in \Delta^{\#L-1}_+ \,:\, p'_s \left(\sum_{h\in H} \left[x^h_s - Y_s x^h_0\right] - W_s \right) > p_s \left(\sum_{h\in H} \left[x^h_s - Y_s x^h_0\right] - W_s \right) \right\}.$$

Given a class $\mathbb{A}_P$ of primitives, we define, for each state of nature $s \in S$, a reaction correspondence $\psi_M^{s,\mathbb{A}_P} : \mathbb{P}_M \times \mathbb{X}_M^H \twoheadrightarrow \Upsilon_M^s(\mathbb{A}_P)$, which associates a vector (π,η) with the set of vectors $r' := r'_{s,\mathbb{A}_P}(1,1,\ldots,1) \in \Upsilon_M^s(\mathbb{A}_P)$ that satisfies

$$\left(r'_{s,\mathbb{A}_P} \sum_{j \in J(\mathbb{A}_P)} p_s A_{s,j} \sum_{h\in H} \theta_j^h - \sum_{k\in\mathbb{A}_P}\sum_{h\in H} \delta_{s,k}^h \right)^2 < \left(r_{s,\mathbb{A}_P} \sum_{j \in J(\mathbb{A}_P)} p_s A_{s,j} \sum_{h\in H} \theta_j^h - \sum_{k\in\mathbb{A}_P}\sum_{h\in H} \delta_{s,k}^h \right)^2,$$

where $\pi = (p,q,r)$ and $(r_{s,j})_{j\in J(\mathbb{A}_P)} := r_{s,\mathbb{A}_P}(1,1,\ldots,1)$. Finally, given a class $\mathbb{A}_C$ of primitives, for each state of nature $s \in S$, and for each $m \in \{1,2,\ldots,n(\mathbb{A}_C)\}$, we define the reaction correspondence

$$\psi_M^{s,j^m(\mathbb{A}_C)} : \mathbb{P}_M \times \mathbb{X}_M^H \twoheadrightarrow [\beta_M^{s,m}(\mathbb{A}_C),1],$$

as the set function that associates, to each vector $(\pi,\eta) \in \mathbb{P}_m \times \mathbb{X}_M^H$, the set of numbers $r' \in [\beta_M^{s,m}(\mathbb{A}_C),1]$ that satisfies

$$\left(r' p_s A_{s,j^m(\mathbb{A}_C)} \sum_{h\in H} \theta^h_{j^m(\mathbb{A}_C)} + \sum_{i=1}^{m-1} r_{s,j^i(\mathbb{A}_C)} p_s A_{s,j^i(\mathbb{A}_C)} \sum_{h\in H} \theta^h_{j^i(\mathbb{A}_C)} - \sum_{k\in\mathbb{A}_C}\sum_{h\in H}\delta^h_{s,k} \right)^2$$
$$< \left(\sum_{i=1}^{m} r_{s,j^i(\mathbb{A}_C)} p_s A_{s,j^i(\mathbb{A}_C)} \sum_{h\in H} \theta^h_{j^i(\mathbb{A}_C)} - \sum_{k\in\mathbb{A}_C}\sum_{h\in H}\delta^h_{s,k} \right)^2.$$

DEFINITION 3. *Given $M \in \mathcal{M}$, an equilibrium for the truncated economy $\mathcal{E}_M$ is a vector*

$$(\overline{\pi},\overline{\eta}) = \Big((\overline{p}_M, \overline{q}_M, \overline{r}_M), (\overline{x}_M^h, \overline{\varphi}_M^h, \overline{\delta}_M^h, \overline{\theta}_M^h)_{h\in H} \Big) \in \mathbb{P}_M \times \mathbb{X}_M^H$$

at which all the reaction correspondences defined above have an empty value.

LEMMA 2. *Given a vector $M \in \mathcal{M}$, if Assumptions 1-3 hold, there exists an equilibrium for the truncated economy $\mathcal{E}_M$.*

PROOF: Observe that from Assumption 1 and Assumption 3, $\dot{B}_M^h(\pi)$ has non-empty values and has open graph. Then, it follows from Assumption 2, that the reaction correspondences $(\psi_M^s)_{s\in S^*}$, $(\psi_M^{s,\mathbb{A}_P})_{\{s\in S,\, \mathbb{A}_P\subset\mathcal{A}_P\}}$, $(\psi_M^h)_{h\in H}$, and $(\psi_M^{s,j})_{\{(s,j)\in S\times J(\mathbb{A}_C),\, \mathbb{A}_C\subset\mathcal{A}_C\}}$ satisfy the assumptions of the Gale-Mas-Colell Fixed Point Theorem (see Gale and Mas-Colell [8, 9]), that is, all correspondences are lower hemicontinuous with convex and open values, have the same domain, and the product of the *image spaces* coincides with the domain. Thus, there exists a vector $(\overline{\pi}_M, \overline{\eta}_M) \in \mathbb{P}_M \times \mathbb{X}_M^H$ such as

- $\psi_M^h(\overline{\pi}_M,\overline{\eta}_M) = \emptyset$ or $\overline{\eta}_M^h \in \psi_M^h(\overline{\pi}_M,\overline{\eta}_M)$, for each agent $h \in H$;
- $\psi_M^0(\overline{\pi}_M,\overline{\eta}_M) = \emptyset$ or $((\overline{p}_M)_0, \overline{q}_M) \in \psi_M^0(\overline{\pi}_M,\overline{\eta}_M)$;
- $\psi_M^s(\overline{\pi}_M,\overline{\eta}_M) = \emptyset$ or $(\overline{p}_M)_s \in \psi_M^s(\overline{\pi}_M,\overline{\eta}_M)$, for each state of nature $s \in S$;
- $\psi_M^{s,\mathbb{A}_P}(\overline{\pi}_M,\overline{\eta}_M) = \emptyset$ or $(\overline{r}_M)_{s,\mathbb{A}_P} \in \psi_M^{s,\mathbb{A}_P}(\overline{\pi}_M,\overline{\eta}_M)$, for each class of primitives $\mathbb{A}_P \subset \mathcal{A}_P$ and state of nature $s \in S$;
- $\psi_M^{s,j^m(\mathbb{A}_C)}(\overline{\pi}_M,\overline{\eta}_M) = \emptyset$ or $(\overline{r}_M)_{s,j^m(\mathbb{A}_C)} \in \psi_M^{s,j^m(\mathbb{A}_C)}(\overline{\pi}_M,\overline{\eta}_M)$, for each state of nature $s \in S$, for each class $\mathbb{A}_C \subset \mathcal{A}_C$ and for each $m \in \{1,2,\ldots,n(\mathbb{A}_C)\}$.

Clearly it is not possible to $\overline{\eta}_M^h \notin B_M^h(\overline{\pi}_M)$, because in this case it would neither be a fixed point, nor an empty value. Moreover, we can not have $\overline{\eta}_M^h \in \psi_M^h(\overline{\pi}_M,\overline{\eta}_M)$ because it contradicts the fact that $\overline{\eta}_M^h \notin \hat{\Psi}^{h,M}((\overline{p}_{-0})_M, \overline{\eta}_M^h)$. Thus, we must have $\psi_M^h(\overline{\pi}_M,\overline{\eta}_M) = \emptyset$, for each agent $h \in H$.

As noted above, $\overline{\eta}_M^h \in B_M^h(\overline{\pi}_M)$. Adding over the agents, it follows from Assumption 1 that

$$(\overline{p}_M)_0 \left[\sum_{h\in H} (\overline{x}_M^h)_0 - W_0 \right] + \sum_{i\in\{P,C\}} \left[\sum_{\mathbb{A}_i\subset\mathcal{A}_i} \left(\sum_{j\in J(\mathbb{A}_i)} (\overline{q}_M)_j \sum_{h\in H} (\overline{\theta}_M^h)_j - \sum_{k\in\mathbb{A}_i} (\overline{q}_M)_k \sum_{h\in H} (\overline{\varphi}_M^h)_k \right) \right] \le 0.$$

Thus, $((\overline{p}_M)_0,\overline{q}_M)\notin\psi^0_M(\overline{\pi}_M,\overline{\eta}_M)$ and, therefore, $\psi^0_M(\overline{\pi}_M,\overline{\eta}_M)$ is an empty set. Finally, one can easily see that, from the definition, the correspondences ψ^s_M, $\psi^{s,\mathbb{A}_P}_M$ and $\psi^{s,j^m(\mathbb{A}_C)}_M$ may not have a fixed point. Then, we must have that $\psi^s_M(\overline{\pi}_M,\overline{\eta}_M)=\emptyset$, $\psi^{s,\mathbb{A}_P}_M(\overline{\pi}_M,\overline{\eta}_M)=\emptyset$ and $\psi^{s,j^m(\mathbb{A}_C)}_M(\overline{\pi}_M,\overline{\eta}_M)=\emptyset$. □

Now, for convenience of notation, when mistakes are not possible, we suppress the subscript of the allocations $(\overline{\pi}_M,\overline{\eta}_M)$. So, using this notation, with $M\in\mathcal{M}$ fixed, we already know that an equilibrium allocation for the truncated economy, $(\overline{\pi},\overline{\eta})$, satisfies $\overline{\eta}^h\in B^h_M(\overline{\pi})$ and $\hat{\Psi}^{h,M}(\overline{p}_{-0},\overline{\eta}^h)\cap\dot{B}^h_M(\overline{\pi})=\emptyset$. Moreover, the fact that both sets $\hat{\Psi}^{h,M}(\overline{p}_{-0},\overline{\eta}^h)$ and $\dot{B}^h_M(\overline{\pi})$ are open implies that $\hat{\Psi}^{h,M}(\overline{p}_{-0},\overline{\eta}^h)\cap \mathit{closure}\,[\dot{B}^h_M(\overline{\pi})]=\emptyset$. As $\dot{B}^h_M(\overline{\pi})$ is non-empty and convex, we conclude that $\hat{\Psi}^{h,M}(\overline{p}_{-0},\overline{\eta}^h)\cap B^h_M(\overline{\pi})=\emptyset$. Furthermore, since $\psi^0(\overline{\pi},\overline{\eta})=\emptyset$, for any $(p',q'_K,q'_J)\in\Xi 1$ we have that

$$p'_0\left[\sum_{h\in H}\overline{x}^h_0-W_0\right]+\sum_{i\in\{P,C\}}\left[\sum_{\mathbb{A}_i\subset\mathcal{A}_i}\left(\sum_{j\in J(\mathbb{A}_i)}q'_j\sum_{h\in H}\overline{\theta}^h_j-\sum_{k\in\mathbb{A}_i}q'_k\sum_{h\in H}\overline{\varphi}^h_k\right)\right]\leq 0. \tag{7}$$

Thus, suppose that $\sum_h\overline{x}^h_{0,l}-W_{0,l}>0$ for some $l\in L$. Then, setting $p'_{0,l}=1$, $p'_{0,l'}=0$ for all $l'\neq l$, $q_J=0$ and $q_K=0$, we obtain a contradiction. Moreover, suppose that $\sum_h\overline{\theta}^h_j>\sum_h\overline{\varphi}^h_k$ for some pair $(k,j)\in\mathbb{A}_i\times J(\mathbb{A}_i)$. Thus, letting $p_0=0$, $q_j=1$, and $q_{j'}=0$ for all $j'\neq j$, $q_k=1$, and $q_{k'}=0$ for all $k'\neq k$, we obtain a contradiction with equation (7). Hence, it follows that $\sum_h\overline{x}^h_{0,l}-W_{0,l}\leq 0$ for $l\in L$, and $\sum_h\overline{\theta}^h_j\leq\sum_h\overline{\varphi}^h_k$ for each pair $(k,j)\in\mathbb{A}_i\times J(\mathbb{A}_i)$, and for all $\mathbb{A}_i\in\mathcal{A}_i$ with $i\in\{P,C\}$.

Lemma 3. *There exists $M_1^\star>0$ such that, for each $M_1>M_1^\star$, if Assumptions 1-3 hold, each equilibrium allocations $(\overline{\pi},\overline{\eta})$ for a truncated economy $\mathcal{E}_M$, with $M=(M_1,M_2)\in\mathcal{M}$, satisfies*

(3.1) *For each agent $h\in H$, $\overline{\eta}^h\in B^h_M(\overline{\pi})$;*
(3.2) $\hat{\Psi}^{h,M}(\overline{p}_{-0},\overline{\eta}^h)\cap B^h_M(\overline{\pi})=\emptyset,\ \ \forall\, h\in H$;
(3.3) $\sum_{h\in H}\overline{x}^h_0=W_0$;
(3.4) $\sum_{j\in J(\mathbb{A}_i)}\overline{q}_j\sum_{h\in H}\overline{\theta}^h_j=\sum_{k\in\mathbb{A}_i}\overline{q}_k\sum_{h\in H}\overline{\varphi}^h_k$, *for each class $\mathbb{A}_i\subset\mathcal{A}_i$ with $i\in\{P,C\}$;*
(3.5) *For each $s\in S$ and $\mathbb{A}_P\subset\mathcal{A}_P$,*

$$\overline{r}_{s,\mathbb{A}_P}\in\arg\max_{r\in[\beta^s_M(\mathbb{A}_P),1]}-\left(r\sum_{j\in J(\mathbb{A}_P)}\overline{p}_sA_{s,j}\sum_{h\in H}\overline{\theta}^h_j-\sum_{k\in\mathbb{A}_P}\sum_{h\in H}\overline{\delta}^h_{s,k}\right)^2;$$

(3.6) *For each $s\in S$, $\mathbb{A}_C\subset\mathcal{A}_C$, $j^m(\mathbb{A}_C)\in J(\mathbb{A}_C)$, the payment rate $\overline{r}_{s,j^m(\mathbb{A}_C)}$ minimizes the function*

$$\left(r\,F^{s,m}_{\mathbb{A}_C}(\overline{\pi},\overline{\eta})+\sum_{i=1}^{m-1}\overline{r}_{s,j^i(\mathbb{A}_C)}F^{s,i}_{\mathbb{A}_C}(\overline{\pi},\overline{\eta})-\sum_{k\in\mathbb{A}_C}\sum_{h\in H}\overline{\delta}^h_{s,k}\right)^2,$$

subject to $r\in[\beta^{s,m}_M(\mathbb{A}_C),1]$, where $F^{s,i}_{\mathbb{A}_C}(\overline{\pi},\overline{\eta}):=\overline{p}_sA_{s,j^i(\mathbb{A}_C)}\sum_{h\in H}\overline{\theta}^h_{j^i(\mathbb{A}_C)}$;
(3.7) *There exists $\mathcal{X}<M_1^\star$ such that, for each $s\in S^*$ and $l\in L$, the consumption allocations $(\overline{x}^h_{s,l})_{h\in H}$ satisfy, $\overline{x}^h_{s,l}\leq W_{s,l}\leq\mathcal{X}$;*
(3.8) *For each $s\in S$ and $l\in L$,*

$$\sum_{h\in H}\overline{x}^h_{s,l}-(Y_sW_0)_l-W_{s,l}\leq\sum_{i\in\{P,C\}}\left[\sum_{\mathbb{A}_i\subset\mathcal{A}_i}\left(\sum_{j\in J(\mathbb{A}_i)}\overline{r}_{s,j}\overline{p}_sA_{s,j}\sum_{h\in H}\overline{\theta}^h_j-\sum_{k\in\mathbb{A}_i}\sum_{h\in H}\overline{\delta}^h_{s,k}\right)\right];$$

(3.9) $\sum_h\overline{\theta}^h_j\leq\sum_h\overline{\varphi}^h_k$ *for each pair $(k,j)\in\mathbb{A}_i\times J(\mathbb{A}_i)$, for all $\mathbb{A}_i\subset\mathcal{A}_i$, with $i\in\{P,C\}$.*

PROOF: As discussed above, items (3.1), (3.2) and (3.9) hold for each $M = (M_1, M_2) \in \mathcal{M}$. Now, as $\sum_h \overline{x}^h_{0,l} - W_{0,l} \leq 0$ for $l \in L$, there exists M'_1 such that, for each $M_1 > M'_1$, an equilibrium consumption allocation of the truncated economy $\mathcal{E}_M$ satisfies $\overline{x}^h_{0,l} < M_1$. Thus, given $M_1 > M'_1$, suppose that agent h equilibrium allocation satisfies

$$\overline{p}_0\overline{x}^h_0 + \sum_{i\in\{P,C\}} \left[\sum_{\mathbb{A}_i\subset\mathcal{A}_i} \left(\sum_{j\in J(\mathbb{A}_i)} \overline{q}_j\overline{\theta}^h_j - \sum_{k\in\mathbb{A}_i} \overline{q}_k\overline{\varphi}^h_k\right)\right] < m^h_0(\overline{p}_0).$$

As $\overline{x}^h_0$ is interior, there exists $\hat{x}^h_0 \gg \overline{x}^h_0$ such that $\hat{\eta}^h = (\hat{x}^h_0, \overline{\varphi}^h, \overline{\delta}^h, \overline{\theta}^h) \in B^h_M(\overline{\pi})$. From the strict monotonicity of $\hat{\Psi}^{h,M}$ on x_0, we have that $\hat{\Psi}^{h,M}(\overline{p}_{-0}, \overline{\eta}^h) \cap B^h_M(\overline{\pi}) \neq \emptyset$, which contradicts item (3.2). Thus, for each agent h, the budget constraint for the first period must hold with equality. Summing over the agents, it follows from Assumption 1 that

$$\overline{p}_0\left[\sum_{h\in H}\overline{x}^h_0 - W_0\right] + \sum_{i\in\{P,C\}} \left[\sum_{\mathbb{A}_i\subset\mathcal{A}_i} \left(\sum_{j\in J(\mathbb{A}_i)} \overline{q}_j \sum_{h\in H}\overline{\theta}^h_j - \sum_{k\in\mathbb{A}_i} \overline{q}_k \sum_{h\in H}\overline{\varphi}^h_k\right)\right] = 0. \tag{8}$$

Now, given a class $\mathbb{A}_i$, defining k' as

$$\sum_{h\in H}\varphi^h_{k'} = \min_{k\in\mathbb{A}_i}\sum_{h\in H}\varphi^h_k,$$

it follows from $\sum_{h\in H}\overline{\theta}^h_j \leq \sum_{h\in H}\overline{\varphi}^h_k$, for all $(j,k) \in J(\mathbb{A}_i)\times\mathbb{A}_i$, that

$$\begin{aligned} \sum_{j\in J(\mathbb{A}_i)} \overline{q}_j \sum_{h\in H}\overline{\theta}^h_j - \sum_{k\in\mathbb{A}_i} \overline{q}_k \sum_{h\in H}\overline{\varphi}^h_k &\leq \sum_{j\in J(\mathbb{A}_i)} \overline{q}_j \sum_{h\in H}\overline{\varphi}^h_{k'} - \sum_{k\in\mathbb{A}_i} \overline{q}_k \sum_{h\in H}\overline{\varphi}^h_{k'} \\ &= \sum_{h\in H}\overline{\varphi}^h_{k'}\left(\sum_{j\in J(\mathbb{A}_i)}\overline{q}_j - \sum_{k\in\mathbb{A}_i}\overline{q}_k\right) \leq 0, \end{aligned} \tag{9}$$

where the last inequality is a consequence of $\sum_{j\in J(\mathbb{A}_i)}\overline{q}_j \leq \sum_{k\in\mathbb{A}_i}\overline{q}_k$. It follows from (9) and from the inequality $\sum_h \overline{x}^h_0 \leq W_0$ that the left hand side of equation (8) is a sum of non-positive terms. Thus, each term must be zero, and condition (3.4) hold, i.e

$$\sum_{j\in J(\mathbb{A}_i)} \overline{q}_j \sum_{h\in H}\overline{\theta}^h_j - \sum_{k\in\mathbb{A}_i} \overline{q}_k \sum_{h\in H}\overline{\varphi}^h_k = 0,$$

for each class $\mathbb{A}_i \subset \mathcal{A}_i$ with $i \in \{P, C\}$. Moreover, suppose that there exists a commodity $l \in L$ such that, $\sum_{h\in H}\overline{x}^h_{0,l} < W_{0,l}$. From equation 8, we must have $\overline{p}_{0,l} = 0$. But it follows from the strict monotonicity of $\hat{\Psi}^{h,M}$ on $x_{0,l}$ that $B^h_M(\overline{\pi}) \cap \hat{\Psi}^{h,M}(\overline{p}_{-0}, \overline{\eta}^h) \neq \emptyset$, which is a contradiction. Therefore, item (3.3) holds.

It follows from Lemma 2 that, given a state of nature $s \in S$, a class of primitives $\mathbb{A}_P \subset \mathcal{A}_P$ and $M = (M_1, M_2) \in \mathcal{M}$, we have, for each $r \in [\beta^s_M(\mathbb{A}_P), 1]$,

$$\left(r\sum_{j\in J(\mathbb{A}_P)}\overline{p}_s A_{s,j}\sum_{h\in H}\overline{\theta}^h_j - \sum_{k\in\mathbb{A}_P}\sum_{h\in H}\overline{\delta}^h_{s,k}\right)^2 \geq \left(\overline{r}_{s,\mathbb{A}_P}\sum_{j\in J(\mathbb{A}_P)}\overline{p}_s A_{s,j}\sum_{h\in H}\overline{\theta}^h_j - \sum_{k\in\mathbb{A}_P}\sum_{h\in H}\overline{\delta}^h_{s,k}\right)^2.$$

Hence,

$$\overline{r}_{s,\mathbb{A}_P} \in \arg\max_{r\in[\beta^s_M(\mathbb{A}_P),1]} -\left(r\sum_{j\in J(\mathbb{A}_P)}\overline{p}_s A_{s,j}\sum_{h\in H}\overline{\theta}^h_j - \sum_{k\in\mathbb{A}_P}\sum_{h\in H}\overline{\delta}^h_{s,k}\right)^2$$

and (3.5) is proved. With analogous arguments, we can guarantee item (3.6).

Furthermore, given an equilibrium $(\overline{\pi}, \overline{\eta})$ for the abstract economy $\mathcal{E}_M$, with $M = (M_1, M_2)$ and $M_1 > M'_1$, we know that $\psi^s_M(\overline{\pi}, \overline{\eta}) = \emptyset$ for each state of nature $s \in S$. Then, for all prices $p'_s \in \Delta^{\#L-1}_+$ we have

$$p'_s\left(\sum_{h\in H}\left[\overline{x}^h_s - Y_s\overline{x}^h_0\right] - W_s\right) \leq \overline{p}_s\left(\sum_{h\in H}\left[\overline{x}^h_s - Y_s\overline{x}^h_0\right] - W_s\right). \tag{10}$$

Moreover, it follows from item (3.1) that $\overline{\eta}^h \in B^h_M(\overline{\pi})$ for each agent $h \in H$. Thus, given an state of nature $s \in S$,

$$(11) \qquad \overline{p}_s \left(\sum_{h\in H} \left[\overline{x}^h_s - Y_s \overline{x}^h_0 \right] - W_s \right) \leq \sum_{i\in\{P,C\}} \left[\sum_{\mathbb{A}_i \subset \mathcal{A}_i} \left(\sum_{j\in J(\mathbb{A}_i)} \overline{r}_{s,j} \overline{p}_s A_{s,j} \sum_{h\in H} \overline{\theta}^h_j - \sum_{k\in\mathbb{A}_i} \sum_{h\in H} \overline{\delta}^h_{s,k} \right) \right].$$

Letting, at equation (10), $p'_{s,l} = 1$ and $p'_{s,l'} = 0$ for each $l' \neq l$, we have from (10), (11) and item (3.3) that

$$(12) \qquad \sum_{h\in H} \overline{x}^h_{s,l} - (Y_s W_0)_l - W_{s,l} \leq \sum_{i\in\{P,C\}} \left[\sum_{\mathbb{A}_i \subset \mathcal{A}_i} \left(\sum_{j\in J(\mathbb{A}_i)} \overline{r}_{s,j} \overline{p}_s A_{s,j} \sum_{h\in H} \overline{\theta}^h_j - \sum_{k\in\mathbb{A}_i} \sum_{h\in H} \overline{\delta}^h_{s,k} \right) \right],$$

which proofs (3.8). As in the economy $\mathcal{E}_M$, (a) the positions on primitives, $\overline{\varphi}^h_j$, are bounded by above by 2Ω and (b) the aggregated purchase of each derivative, $\sum_{h\in H} \overline{\theta}^h_j$, is bounded by the total short position on primitives; it follows from equation (12) that

$$\sum_{h\in H} \overline{x}^h_{s,l} - (Y_s W_0)_l - W_{s,l} \leq 2 \sum_{j\in J} \|A_{s,j}\|_1 \, (\#H) \, \Omega.$$

Then, for each $l \in L$,

$$\overline{x}^h_{s,l} \leq \max_{(s,l)\in S\times L} \left\{ W_{s,l} + (Y_s W_0)_l + 2 \sum_{j\in J} \|A_{s,j}\|_1 \, (\#H) \, \Omega \right\}, \quad \forall h \in H,$$

which guarantees that consumption allocations $\overline{x}^h_s$, $s \in S$, are uniformly bounded from above, independently of the value of $M_1 > M'_1$. Moreover, item (3.3) guarantees that first period consumption allocations, $\overline{x}^h_0$, are also uniformly bounded, independent of $M = (M_1, M_2)$. Therefore, there exists $\mathcal{X} > 0$ and $M^\star_1 > \max\{\mathcal{X}, M'_1\}$ such that, $\overline{x}^h_{s,l} \leq W_{s,l} \leq \mathcal{X} < M^\star_1$, for all $(s,l) \in S^* \times L$, which proofs item (3.7). □

DEFINITION 4. *Given $M \in \mathcal{M}$, a M-semi-equilibrium is an allocation $(\tilde{\pi}_M, \tilde{\eta}_M) \in \mathbb{P}_M \times \mathbb{X}^H_M$ which satisfies items (3.1)-(3.8).*

It is important to remark that item (3.9) does not enter into the definition of M-semi-equilibrium. Note that, given $M = (M_1, M_2)$, it follows from Lemma 3 that, for a given $M_1 > M^\star_1$, an M-semi-equilibrium always exists . For convenience of notation, we also suppress the subscript M on M-semi-equilibrium allocations when mistakes are not possible.

LEMMA 4. *There exists $M^{\star\star}_1 > M^\star_1$ such that, if Assumptions 1-4 hold, for each M-semi-equilibrium $(\tilde{\pi}, \tilde{\eta})$, with $M = (M_1, M_2)$ and $M_1 > M^{\star\star}_1$, the commodity prices $\tilde{p}_{s,l}$, with $(s,l) \in S^* \times L$, have a uniform lower bound $\underline{p}$, strictly greater than zero and independent of $M = (M_1, M_2)$.*

PROOF: Fix $M = (M_1, M_2)$ with $M_1 > M^\star_1$. It follows from (3.7) that, a M-semi-equilibrium allocation $(\tilde{\pi}, \tilde{\eta})$ satisfies $\tilde{x}^h_{s,l} \leq \mathcal{X} < M_1$, for all $(s,l) \in S^* \times L$, which guarantees that $\tilde{p}_{s,l} > 0$.[9] Moreover, it follows

[9]In other case, as preferences are strictly monotonic on consumption, each agent $h \in H$ could increase the consumption of a zero-price commodity, choosing another allocation $\hat{\eta}^h$ that improves their situation and still belongs to the budget set $B^h_M(\tilde{\pi})$, which contradicts item (3.2).

from Assumption 1 that, for each $s \in S$ and each $h \in H$, $\underline{m}_s^h := \min_{p_s \in \Delta_+^{\#L-1}} m_s^h(p_s) > 0$, because $\Delta_+^{\#L-1}$ is compact. Also, defining, for each pair of different commodities (l,l'), the compact set

$$G(l,l') = \left\{ p_0 \in \mathbb{R}_+^L \,:\, \left(p_{0,l'} \geq \frac{1-p_{0,l}}{\#L+\#J-1} \right) \wedge \left(\exists (q_K, q_J), (p_0, q_K, q_J) \in \Xi \right) \right\},$$

we have that, since $p_0 = 0$ does not belong to $G(l,l')$,

$$\underline{m}_0^h := \min_{l \in L} \min_{l' \neq l} \min_{p_0 \in G(l,l')} m_0^h(p_0) > 0, \quad \forall h \in H.$$

Given a M-semi-equilibrium $(\tilde{\pi}, \tilde{\eta})$ the vector $(\tilde{p}_0, \tilde{q}_J) \in \Delta_+^{\#L+\#J-1}$. Therefore, for a fixed $l \in L$, in order to guarantee that $\tilde{p}_{0,l}$ is uniformly bounded independent of M, we have to consider two possibilities, as in Seghir and Torres-Martínez (2004):

CASE I: *There exists a commodity $l' \neq l$ for which $\tilde{p}_{0,l'} \geq \frac{1-\tilde{p}_{0,l}}{\#L+\#J-1}$.*

In this case, $\tilde{p}_0 \in G(l,l')$, which implies that $m_0^h(\tilde{p}_0) \geq \underline{m}_0^h$. Thus, any agent h can choose the allocation $(\hat{x}^h, 0, 0, 0)$, defined by

$$\hat{x}^h_{s'',l''} = \begin{cases} \epsilon & , \text{ if } (s'',l'') \neq (0,l), \\ \min\left\{ \frac{\underline{m}_0^h}{2\tilde{p}_{0,l}}, M_1 \right\} & , \text{ if } (s'',l'') = (0,l), \end{cases}$$

where $\epsilon := \min_{h \in H} \left\{ \frac{\underline{m}_0^h}{2} \,;\, \min_{s \in S} \frac{\underline{m}_s^h}{2} \right\} > 0$.

On the other hand, since each consumption allocation $(\tilde{x}^h)_{h \in H}$ is uniformly bounded, it follows from Assumption 4 that

$$Z_{0,l}^h(\tilde{x}, \tilde{d}, \epsilon) \leq \tilde{Z}_\epsilon := \max_{h \in H} \max_{(s'',l'') \in S^* \times L} Z_{s'',l''}^h \left((\mathcal{X}, \mathcal{X}, \ldots, \mathcal{X}), 0, \epsilon \right).$$

As the left hand side in the inequality above does not depends on M, there exists $(M_1^\star)' \geq M_1^\star$ such that, if $M_1 > (M_1^\star)'$, $\tilde{Z}_\epsilon < M_1$. Thus, it follows from Assumption 4 and from the optimality condition (3.2) that, for each fixed M-semi-equilibrium with $M_1 > (M_1^\star)'$, $\tilde{Z}_\epsilon > \frac{\underline{m}_0^h}{2\tilde{p}_{0,l}}$, which implies that

$$\tilde{p}_{0,l} \geq \underline{p}_0^I := \max_{h \in H} \frac{\underline{m}_0^h}{2\tilde{Z}_\epsilon} > 0.$$

CASE II: *There exists an asset $j \in J$ for which $\tilde{q}_j \geq \frac{1-\tilde{p}_{0,l}}{\#L+\#J-1}$.*

Define $\underline{W}_0 = \min_{l \in L} W_{0,l}$. Note that, there always exists an agent $h(\tilde{p}_0) \in H$ that can demand $\frac{\underline{W}_0}{\#H}$ units of each good at the first period, without making any financial transaction. In fact, suppose that such agent does not exist. Then, it follows from the first period budget constraint that $m_0^h(\tilde{p}_0) < \|\tilde{p}_0\|_1 \frac{\underline{W}_0}{\#H}$ for all $h \in H$. Assumption 1, however, implies that $\sum_{h \in H} m_0^h(\tilde{p}_0) \geq \|\tilde{p}_0\|_1 \underline{W}_0$, which is a contradiction. Moreover, since we are restricting $(p_0, q_K, q_J) \in \Xi$, it follows that there exists $k \in K$ for which $\tilde{q}_k \geq \frac{1-\tilde{p}_{0,l}}{(\#L+\#J-1)\#K}$.

Now, the agent $h(\tilde{p}_0)$ can demand the bundle $\hat{x}^{h(\tilde{p}_0)}$, defined as

$$\hat{x}^{h(\tilde{p}_0)}_{s',l'} = \begin{cases} \epsilon' & , \text{ if } (s',l') \neq (0,l), \\ \min\left\{ \epsilon' + \frac{q_k \gamma}{\tilde{p}_{0,l}}, M_1 \right\} & , \text{ if } (s',l') = (0,l), \end{cases}$$

where $\epsilon' := \min_{h \in H} \left\{ \frac{\underline{W}_0}{2\#H} \,;\, \min_{s \in S} \frac{\underline{m}_s^h}{2} \right\} > 0$, selling γ units of the primitive k, without making any other financial transaction, and paying all his promises at the second period, where γ satisfy

$$\gamma \left(\max_{(p_0,q_k) \in \Xi_k} \|C_{k,l}(p_0,q_k)\|_1 \right) \leq \frac{\underline{W}_0}{2\#H}; \qquad \gamma \leq 2\Omega(\#H); \qquad \gamma A_{s,k} \leq \epsilon', \ \forall s \in S.$$

Therefore, this allocation belongs to the budget set of agent $h(\tilde{p}_0)$ and γ is independent on prices. Hence, it follows from Assumption 4 and from optimality condition (3.2) that there exists $(M_1^\star)_0 > (M_1^\star)'$ such that, for each fixed M-semi-equilibrium, with $M = (M_1, M_2)$, if $M_1 > (M_1^\star)_0$ then $\epsilon' + \frac{\tilde{q}_k \gamma}{\tilde{p}_{0,l}} \leq \tilde{Z}_{\epsilon'}$, which implies that

$$\tilde{p}_{0,l} \geq \underline{p}_0^{II} := \frac{\gamma}{\gamma + (\#L + \#J - 1)\#K\, \tilde{Z}_{\epsilon'}}.$$

Therefore, Cases I and II imply that the commodity M-semi-equilibrium prices for the first period (where $M_1 > (M_1^\star)_0$) are uniformly bounded from below by $\tilde{p}_{0,l} \geq \underline{p}_0 := \min\{\underline{p}_0^I\,;\, \underline{p}_0^{II}\}$.

Now, since $\tilde{p}_{0,l} \geq \underline{p}_0$, define ϵ_S as

$$\epsilon_S := \min_{h \in H} \left\{ \min_{p_0 \in \Xi_1} m_0^h(p_0);\, \min_{s \in S} \underline{m}_s^h \right\} > 0,$$

where Ξ_1 denotes the set of prices $p_0 \geq \underline{p}_0(1, 1, \ldots, 1)$ such that there exists prices q for which $(p_0, q) \in \Xi$.

Thus, for a given M-semi-equilibrium $(\tilde{\pi}, \tilde{\eta})$, with $M_1 > (M_1^\star)_0$, and for a fixed pair $(s, l) \in S \times L$, any agent can demand an allocation $(\hat{x}^h, 0, 0, 0)$, defined as

$$\hat{x}^h_{s',l'} = \begin{cases} \epsilon_S & ,\ \text{if}\,(s', l') \neq (s, l), \\ \min\left\{ \frac{\underline{m}_s^h}{2\tilde{p}_{s',l'}}, M_1 \right\} & ,\ \text{if}\,(s', l') = (s, l). \end{cases}$$

Then, there exist $M_1^{\star\star} > \max\{\tilde{Z}_{\epsilon_S}, (M_1^\star)_0\}$ such that it follows from Assumption 4 and from the optimality condition (3.2) that, if $M_1 > M_1^{\star\star}$, then $\tilde{Z}_{\epsilon_S} > \frac{\underline{m}_s^h}{2\tilde{p}_{s',l'}}$. This implies that the commodity M-semi-equilibrium prices at the second period are uniformly bounded from below by

$$\tilde{p}_{s,l} \geq \underline{p}_s := \max_{h \in H} \frac{\underline{m}_s^h}{2\tilde{Z}_{\epsilon_S}} > 0.$$

Therefore, we conclude that, for each M-semi-equilibrium $(\tilde{\pi}, \tilde{\eta})$ with $M_1 > M_1^{\star\star}$, the commodity prices $(\tilde{p}_s)_{s \in S^*}$ satisfy $\tilde{p}_{s,l} \geq \underline{p} := \min_{s \in S^*} \underline{p}_s$. □

Now, take $M = (M_1, M_2) \in \mathcal{M}$ such that $M_1 > M_1^{\star\star}$. Fix an M-semi-equilibrium allocation $(\breve{\pi}, \breve{\eta})$ that also satisfies item (3.9) (note that, it is sufficient to take an equilibrium of the truncated economy $\mathcal{E}_M$). Given a class of primitives $\mathbb{A}_i$, with $i \in \{P, C\}$, it follows from items (3.4) and (3.9) that, if there exists $j' \in J(\mathbb{A}_P)$ such that $\sum_{h \in H} \breve{\theta}^h_{j'} < \max_{j \in J(\mathbb{A}_i)} \sum_{h \in H} \breve{\theta}^h_j$, then $\breve{q}_{j'} = 0$. Optimality condition on agents allocations (item (3.2)) implies that, for such j', $\breve{r}_{s,j'} \breve{p}_s A_{s,j'} = 0$ for all $s \in S$. However, as (i) the payment rate of j' is bounded from below by $\frac{1}{M_2} > 0$, and (ii) the commodity prices, at each state $s \in S$, are strictly positive; we must have that $\|A_{s,j'}\|_1 = 0$ for all $s \in S$, which is a contradiction with Assumption 5. Therefore, $\sum_{h \in H} \breve{\theta}^h_{j'} = \sum_{h \in H} \breve{\theta}^h_j$ for all $j, j' \in J(\mathbb{A}_i)$, $\mathbb{A}_i \subset \mathcal{A}_i$ with $i \in \{P, C\}$. Analogously, if there exists a primitive $k \in \mathbb{A}_i$ that satisfies $\sum_{h \in H} \breve{\varphi}^h_k > \min_{k' \in \mathbb{A}_i} \sum_{h \in H} \breve{\varphi}^h_{k'}$, then $\breve{q}_k = 0$.

Thus, it follows from item (3.4) that

$$\sum_{j \in J} \breve{q}_j \sum_{h \in H} \breve{\theta}^h_j = \sum_{k \in \mathbb{A}_i} \breve{q}_k \min_{k' \in \mathbb{A}_i} \sum_{h \in H} \breve{\varphi}^h_{k'},$$

which implies that $\sum_{h \in H} \breve{\theta}^h_j = \min_{k' \in \mathbb{A}_i} \sum_{h \in H} \breve{\varphi}^h_{k'}$ for all j in $J(\mathbb{A}_i)$. Therefore, $\sum_{h \in H} \breve{\theta}^h_j = \sum_{h \in H} \breve{\varphi}^h_k$, for all pair $(k, j) \in \mathbb{A}_i \times J(\mathbb{A}_i)$ such that the M-semi-equilibrium price $\breve{q}_k$ is strictly positive.

Define a new allocation $(\tilde{\pi},\tilde{\eta}) \in \mathbb{P}_M \times \mathbb{X}_M^H$ as

$$\left(\tilde{\pi};\, \tilde{x}^h, \tilde{\delta}^h, \tilde{\theta}_j^h\right) = \left(\breve{\pi};\, \breve{x}^h, \breve{\delta}^h, \breve{\theta}_j^h\right), \ \ \forall j \in J;$$

$$\tilde{\varphi}_k^h = \begin{cases} \breve{\varphi}_k^h, & \text{if } \breve{q}_k > 0; \\ 0\,, & \text{if } \breve{q}_k = 0. \end{cases} \qquad \forall h \in H,\, \forall k \in K;$$

It follows that the allocation $(\tilde{\pi},\tilde{\eta})$ is still a M-semi-equilibrium. Moreover, for a given class $\mathbb{A}_i \subset \mathcal{A}_i$ with $i \in \{P, C\}$, the following conditions are satisfied,

$$\text{(13)} \qquad \sum_{h\in H} \tilde{\theta}_j^h = \sum_{h\in H} \tilde{\varphi}_k^h, \ \ \forall\, (k,j) \in \mathbb{A}_i \times J(\mathbb{A}_i), \text{ for which } \ \tilde{q}_k > 0;$$

$$\text{(14)} \qquad \sum_{h\in H} \tilde{\varphi}_k^h = 0, \ \ \forall\, k \in K, \text{ for which } \ \tilde{q}_k = 0.$$

LEMMA 5. *There exists $M_1^{\star\star\star} > M_1^{\star\star}$ such that, for each $M = (M_1, M_2)$ with $M_1 > M_1^{\star\star\star}$, there exists a M-semi-equilibrium $(\tilde{\pi},\tilde{\eta})$ in which each class of primitives $\mathbb{A}_P \subset \mathcal{A}_P$ has associated anonymous payment rates $(\tilde{r}_{s,j})_{j\in J(\mathbb{A}_P),\, s\in S}$ that satisfy (13), (14) and*

(5.1) *$\tilde{r}_{s,j} = \tilde{r}_{s,j'}$, for all $j, j' \in J(\mathbb{A}_P)$, for all $s \in S$;*

(5.2)

$$0 \le \sum_{j\in J(\mathbb{A}_P)} \tilde{r}_{s,j} \tilde{p}_s A_{s,j} \sum_{h\in H} \tilde{\theta}_j^h - \sum_{k\in\mathbb{A}_P} \sum_{h\in H} \tilde{\delta}_{s,k}^h \le \frac{2}{M_2} \sum_{j\in J(\mathbb{A}_P)} \|A_{s,j}\|_1 (\#H)^2\, \Omega.$$

PROOF: We already know that, for each $M = (M_1, M_2)$ with $M_1 > M_1^{\star\star}$, there exists a M-semi-equilibrium that satisfies equations (13) and (14). Then, for a given M with $M_1 > M_1^{\star\star}$, fix an M-semi-equilibrium in which (13) and (14) hold. Given $s \in S$ and $\mathbb{A}_P \subset \mathcal{A}_P$, it follows from the fact that $(\tilde{r}_{s,j})_{j\in J(\mathbb{A}_P)} \in \Upsilon_M^s(\mathbb{A}_P)$ that

$$\tilde{r}_{s,j} = \tilde{r}_{s,j'}, \ \ \forall j, j' \in J(\mathbb{A}_P), \forall s \in S,$$

which proves item (5.1). As the value of $\tilde{r}_{s,j}$ is independent from $j \in J(\mathbb{A}_P)$ we will denote it by $\tilde{r}_{s,\mathbb{A}_P}$.

Note that equations (13) and (14), jointly with Assumption 5, imply that

$$\text{(15)} \qquad \begin{aligned} \sum_{k\in\mathbb{A}_P} \sum_{h\in H} \tilde{\delta}_{s,k}^h &= \sum_{\{k\in\mathbb{A}_P:\, \tilde{q}_k \neq 0\}} \sum_{h\in H} \tilde{\delta}_{s,k}^h \le \sum_{\{k\in\mathbb{A}_P:\, \tilde{q}_k \neq 0\}} \tilde{p}_s A_{s,k} \sum_{h\in H} \tilde{\varphi}_k^h \\ &= \tilde{p}_s \sum_{\{k\in\mathbb{A}_P:\, \tilde{q}_k \neq 0\}} A_{s,k} \max_{j\in J(\mathbb{A}_P)} \sum_{h\in H} \tilde{\theta}_j^h \le \tilde{p}_s \sum_{j\in J(\mathbb{A}_P)} A_{s,j} \sum_{h\in H} \tilde{\theta}_j^h. \end{aligned}$$

Moreover, in order to prove item (5.2), we must consider two situations. First, if $\sum_{h\in H} \tilde{\varphi}_k^h = 0$ for all $k \in \mathbb{A}_P$, it follows from item (3.4) and the fact that $\tilde{q}_j > 0$ for $j \in J$ that $\sum_{h\in H} \tilde{\theta}_j = 0$ for all $j \in J(\mathbb{A}_P)$. Thus, for any $r \in [\beta_M^s(\mathbb{A}_P), 1]$, we have that

$$r \sum_{j\in J(\mathbb{A}_P)} \tilde{p}_s A_{s,j} \sum_{h\in H} \tilde{\theta}_j^h - \sum_{k\in\mathbb{A}_P} \sum_{h\in H} \tilde{\delta}_{s,k}^h = 0, \ \ \forall s \in S,$$

and, as a particular case, $\tilde{r}_{s,\mathbb{A}_P}$ satisfies the above equation. So, whenever primitives in $\mathbb{A}_P$ are not negotiated (5.2) holds.

On the other hand, if $\sum_{h\in H} \tilde{\varphi}_k^h > 0$ for some $k \in \mathbb{A}_P$, we have to analyze two sub-cases:

Case I: Suppose that $\min_{k\in\mathbb{A}_P}\{||A_{s,k}||_1,\,\overline{c}_{s,k}\}=0$, then $\beta^s_M(\mathbb{A}_P)=\frac{1}{M_2}$ and, since $(\tilde{\pi},\tilde{\eta})$ is a M-semi-equilibrium, we know that

$$\tilde{r}_{s,\mathbb{A}_P}\in\arg\max_{r\in[\frac{1}{M_2},1]}-\left(r\sum_{j\in J(\mathbb{A}_P)}\tilde{p}_sA_{s,j}\sum_{h\in H}\tilde{\theta}^h_j-\sum_{k\in\mathbb{A}_P}\sum_{h\in H}\tilde{\delta}^h_{s,k}\right)^2.$$

Therefore, it follows from (15) that

$$\tilde{r}_{s,\mathbb{A}_P}\sum_{j\in J(\mathbb{A}_P)}\tilde{p}_sA_{s,j}\sum_{h\in H}\tilde{\theta}^h_j\geq\sum_{k\in\mathbb{A}_P}\sum_{h\in H}\tilde{\delta}^h_{s,k}. \tag{16}$$

Thus, if $\frac{1}{M_2}\sum_{j\in J(\mathbb{A}_P)}\tilde{p}_sA_{s,j}\sum_{h\in H}\tilde{\theta}^h_j\leq\sum_{k\in\mathbb{A}_P}\sum_{h\in H}\tilde{\delta}^h_{s,k}$, then

$$\tilde{r}_{s,\mathbb{A}_P}\sum_{j\in J(\mathbb{A}_P)}\tilde{p}_sA_{s,j}\sum_{h\in H}\tilde{\theta}^h_j-\sum_{k\in\mathbb{A}_P}\sum_{h\in H}\tilde{\delta}^h_{s,k}=0. \tag{17}$$

Otherwise, if $\frac{1}{M_2}\sum_{j\in J(\mathbb{A}_P)}\tilde{p}_sA_{s,j}\sum_{h\in H}\tilde{\theta}^h_j>\sum_{k\in\mathbb{A}_P}\sum_{h\in H}\tilde{\delta}^h_{s,k}$, we have that $\tilde{r}_{s,\mathbb{A}_P}=\frac{1}{M_2}$, which implies, jointly with (16) and (17), that

$$\begin{aligned}0\leq\tilde{r}_{s,\mathbb{A}_P}\sum_{j\in J(\mathbb{A}_P)}\tilde{p}_sA_{s,j}\sum_{h\in H}\tilde{\theta}^h_j-\sum_{k\in\mathbb{A}_P}\sum_{h\in H}\tilde{\delta}^h_{s,k}&\leq\frac{1}{M_2}\sum_{j\in J(\mathbb{A}_P)}\tilde{p}_sA_{s,j}\sum_{h\in H}\tilde{\theta}^h_j\\&\leq\frac{2}{M_2}\sum_{j\in J(\mathbb{A}_P)}\|A_{s,j}\|_1(\#H)^2\,\Omega,\end{aligned}$$

which guarantees that item (5.2) holds whenever $\min_{k\in\mathbb{A}_P}\{||A_{s,k}||_1,\,\overline{c}_{s,k}\}=0$ and $\sum_{h\in H}\tilde{\varphi}^h_k>0$ for some $k\in\mathbb{A}_P$.

Case II: If $\min_{k\in\mathbb{A}_P}\{||A_{s,k}||_1,\,\overline{c}_{s,k}\}>0$, we have that $\beta^s_M(\mathbb{A}_P)=\frac{1}{M_1}$. It follows from Assumption 3 that, for each $k\in\mathbb{A}_P$, one of the following two conditions are satisfied:

a. $Y_sC_k(p_0,q_k)\neq0$ for all $(p_0,q_k)\in\Xi_k$ and for each $s\in S$;
b. $C_k(p_0,q_k)=C_k$ for all $(p_0,q_k)\in\Xi_k$.

When item (b) holds, we have that $\overline{c}_{s,k}=\underline{c}_{s,k}:=\min_{(p_0,q_k)\in\Xi_k}\|Y_sC_k(p_0,q_k)\|_1$ for each $s\in S$. On the other hand, since we restrict (p_0,q_k) to the compact set Ξ_k, if item (a) holds, then $\underline{c}_{s,k}>0$. This implies that, given a class $\mathbb{A}_i$ and a state s, $\min_{k\in\mathbb{A}_P}\{||A_{s,k}||_1,\,\overline{c}_{s,k}\}>0$ if and only if $\min_{k\in\mathbb{A}_P}\left\{||A_{s,k}||_1,\,\underline{c}_{s,k}\right\}>0$.

Thus, it follows from Lemma 4 that

$$\begin{aligned}\sum_{k\in\mathbb{A}_P}\sum_{h\in H}\tilde{\delta}^h_{s,k}&\geq\sum_{k\in\mathbb{A}_P}\min\{\tilde{p}_sA_{s,k},\tilde{p}_sY_sC_k(\tilde{p}_0,\tilde{q}_k)\}\sum_{h\in H}\tilde{\varphi}^h_k\geq\underline{p}\sum_{k\in\mathbb{A}_P}\min\left\{\|A_{s,k}\|_1,\underline{c}_{s,k}\right\}\sum_{h\in H}\tilde{\varphi}^h_k\\&\geq\underline{p}\sum_{k\in\mathbb{A}_P:\,\tilde{q}_k\neq0}\min\left\{\|A_{s,k}\|_1,\underline{c}_{s,k}\right\}\max_{j\in J(\mathbb{A}_P)}\sum_{h\in H}\tilde{\theta}^h_j\\&\geq\underline{p}\min_{k\in\mathbb{A}_P}\left\{\|A_{s,k}\|_1,\underline{c}_{s,k}\right\}\max_{j\in J(\mathbb{A}_P)}\sum_{h\in H}\tilde{\theta}^h_j.\end{aligned}$$

Moreover, we know that $\varsigma^s(\mathbb{A}_P):=\underline{p}\min_{k\in\mathbb{A}_p}\left\{\|A_{s,k}\|_1,\underline{c}_{s,k}\right\}$ is strictly positive and it follows from Assumption 5 that $\sum_{j\in J(\mathbb{A}_P)}\|A_{s,j}\|_1>0$. Therefore, there exists $M_1^{\star\star\star}(\mathbb{A}_P)>M_1^{\star\star}$, such that if $M\in\mathcal{M}$ with $M_1>M_1^{\star\star\star}(\mathbb{A}_P)$, we have that $\frac{1}{M_1}\sum_{j\in\mathbb{A}_P}||A_{s,j}||_1\leq\varsigma^s(\mathbb{A}_P)$. Then,

$$\frac{1}{M_1}\tilde{p}_s\sum_{j\in\mathbb{A}_P}A_{s,j}\sum_{h\in H}\tilde{\theta}^h_j\leq\varsigma^s(\mathbb{A}_P)\max_{j\in J(\mathbb{A}_P)}\sum_{h\in H}\tilde{\theta}^h_j\leq\sum_{k\in\mathbb{A}_P}\sum_{h\in H}\tilde{\delta}^h_{s,k}\leq\tilde{p}_s\sum_{j\in\mathbb{A}_P}A_{s,j}\sum_{h\in H}\tilde{\theta}^h_j,$$

which guarantees that the global maximum of $-\left(r\sum_{j\in J(\mathbb{A}_P)}\overline{p}_sA_{s,j}\sum_{h\in H}\overline{\theta}^h_j-\sum_{k\in\mathbb{A}_P}\sum_{h\in H}\overline{\delta}^h_{s,k}\right)^2$ is attainable in this case. This implies that $\tilde{r}_{s,\mathbb{A}_P}\sum_{j\in J(\mathbb{A}_P)}\tilde{p}_sA_{s,j}\sum_{h\in H}\tilde{\theta}^h_j-\sum_{k\in\mathbb{A}_P}\sum_{h\in H}\tilde{\delta}^h_{s,k}=0$. If we take $M_1^{\star\star\star}=\max_{\mathbb{A}_P\subset\mathcal{A}_P}M_1^{\star\star\star}(\mathbb{A}_P)$, then item (5.2) holds whenever $\min_{k\in\mathbb{A}_P}\{||A_{s,k}||_1,\,\overline{c}_{s,k}\}>0$ and

$\sum_{h\in H}\tilde{\varphi}^h_k > 0$ for some $k \in \mathbb{A}_P$. This concludes the proof of Lemma 5. □

For any M-semi-equilibrium $(\breve{\pi}, \breve{\eta})$, consider any allocation $(\breve{\pi}', \breve{\eta}')$ defined by

$$(\breve{p}', \breve{q}', (\breve{r}'_{s,\mathbb{A}_P})_{\{s\in S, \mathbb{A}_P\subset\mathcal{A}_P\}}, \breve{\eta}') = (\breve{p}, \breve{q}, (\breve{r}_{s,\mathbb{A}_P})_{\{s\in S, \mathbb{A}_P\subset\mathcal{A}_P\}}, \breve{\eta})$$

and, for each class $\mathbb{A}_C \subset \mathcal{A}_C$,

$$\breve{r}'_{s,j^m(\mathbb{A}_C)} = \begin{cases} \beta^{s,m}_M(\mathbb{A}_C) & \text{if } \sum_{h\in H}\theta^h_{j^m(\mathbb{A}_C)} = 0, \\ \alpha(j^m(\mathbb{A}_C)) & \text{if } \sum_{h\in H}\theta^h_{j^m(\mathbb{A}_C)} \neq 0 \wedge \|A_{s,j^m(\mathbb{A}_C)}\|_1 = 0, \\ \breve{r}_{s,j^m(\mathbb{A}_C)} & \text{otherwise,} \end{cases} \tag{18}$$

where $\alpha(j^m(\mathbb{A}_C)) \in [\beta^{s,m}_M(\mathbb{A}_C), 1]$. It follows that $B_M(\breve{\pi}') \subset B_M(\breve{\pi})$. Thus, the fact that $\breve{r}_{s,j}$ appears multiplied by $A_{s,j}$ and $\sum_{h\in H}\breve{\theta}_j$ at item (3.6) implies that any $(\breve{\pi}', \breve{\eta}')$ is also a M-semi-equilibrium.

LEMMA 6. *There exists $M^\star_1 > M^{\star\star\star}_1$ such that for each $M = (M_1, M_2) \in \mathcal{M}$, with $M_1 > M^\star_1$, there exists a M-semi-equilibrium $(\tilde{\pi}, \tilde{\eta})$ in which conditions (5.1) and (5.2) hold, and for each class of primitives $\mathbb{A}_C \subset \mathcal{A}_C$ we have*

(6.1) *The rates of payment $(\tilde{r}_{s,j})_{j\in J(\mathbb{A}_C)} \in \mathcal{R}^s_M(\mathbb{A}_C)$, for all $s \in S$, where*[10]

$$\mathcal{R}^s_M(\mathbb{A}_C) \equiv \left\{ r \in \Upsilon^s_M(\mathbb{A}_C) : \exists\, r' \in \mathcal{R}(\mathbb{A}_C), \quad r_m = \max\{ r'_m, \beta^{s,m}\} \right\};$$

(6.2)

$$0 \leq \sum_{j\in J(\mathbb{A}_C)} \tilde{r}_{s,j}\tilde{p}_s A_{s,j} \sum_{h\in H}\tilde{\theta}^h_j - \sum_{k\in\mathbb{A}_C}\sum_{h\in H}\tilde{\delta}^h_{s,k} \leq \frac{2}{M_2}\sum_{j\in J(\mathbb{A}_P)} \|A_{s,j}\|_1 (\#H)^2\,\Omega.$$

PROOF: Given $M = (M_1, M_2) \in \mathcal{M}$, with $M_1 > M^{\star\star\star}_1$, take a M-semi-equilibrium $(\breve{\pi}, \breve{\eta})$ that satisfies the properties (5.1) and (5.2) and equations (13) and (14). We know that such a M-semi-equilibrium exists from Lemma 5. Thus, consider a different allocation $(\tilde{\pi}, \tilde{\eta})$ with

$$(\tilde{p}, \tilde{q}, \tilde{r}_{s,\mathbb{A}_P}, \tilde{\eta})_{\{s\in S, \mathbb{A}_P\subset\mathcal{A}_P\}} = (\breve{p}, \breve{q}, (\breve{r}_{s,\mathbb{A}_P})_{\{s\in S, \mathbb{A}_P\subset\mathcal{A}_P\}}, \breve{\eta})$$

and

$$\tilde{r}_{s,j^m(\mathbb{A}_C)} = \begin{cases} \beta^{s,m}_M(\mathbb{A}_C) & \text{if } \sum_{h\in H}\theta^h_{j^m(\mathbb{A}_C)} = 0; \\ \breve{r}_{s,j^m(\mathbb{A}_C)} & \text{if } \sum_{h\in H}\theta^h_{j^m(\mathbb{A}_C)} \neq 0 \wedge \|A_{s,j^m(\mathbb{A}_C)}\|_1 \neq 0; \\ 1 & \text{if } \sum_{h\in H}\theta^h_{j^m(\mathbb{A}_C)} \neq 0 \wedge \|A_{s,j^m(\mathbb{A}_C)}\|_1 = 0 \wedge m = 1; \\ \tilde{r}_{s,j^{m-1}(\mathbb{A}_C)} & \text{if } \sum_{h\in H}\theta^h_{j^m(\mathbb{A}_C)} \neq 0 \wedge \|A_{s,j^m(\mathbb{A}_C)}\|_1 = 0 \wedge m \neq 1 \wedge \tilde{r}_{s,j^{m-1}(\mathbb{A}_C)} = 1; \\ \beta^{s,m}_M(\mathbb{A}_C) & \text{if } \sum_{h\in H}\theta^h_{j^m(\mathbb{A}_C)} \neq 0 \wedge \|A_{s,j^m(\mathbb{A}_C)}\|_1 = 0 \wedge m \neq 1 \wedge \tilde{r}_{s,j^{m-1}(\mathbb{A}_C)} \neq 1. \end{cases} \tag{19}$$

Since $(\tilde{\pi}, \tilde{\eta})$ respects equation (18), $(\tilde{\pi}, \tilde{\eta})$ is a M-semi-equilibrium and it still satisfies the properties (5.1) and (5.2) of Lemma 5 as well as equations (13) and (14). We will show that $(\tilde{\pi}, \tilde{\eta})$ satisfies all conditions of this lemma.

Fix a class $\mathbb{A}_C \subset \mathcal{A}_C$. We have two cases,

CASE I: *Suppose that $\sum_{h\in H}\tilde{\varphi}^h_k = 0$ for all $k \in \mathbb{A}_C$.*

[10]Equivalently, the set $\mathcal{R}^s_M(\mathbb{A}_C)$ can be defined as

$$\mathcal{R}^s_M(\mathbb{A}_C) \equiv \left\{ r \in \Upsilon^s_M(\mathbb{A}_C) : \exists m,\, 1 \leq m \leq n(\mathbb{A}_C),\, \left(r_{m'} = 1, \forall m' < m\right) \wedge \left(r_{m'} = \beta^{s,m}(\mathbb{A}_C), \forall m' > m\right)\right\}.$$

It follows from item (3.4), and the fact that $\tilde{q}_j > 0$ for $j \in J$, that $\sum_{h\in H} \tilde{\theta}_j = 0$ for all $j \in J(\mathbb{A}_C)$. Thus, for any $(r_{s,j})_{j\in J(\mathbb{A}_C)} \in \Upsilon^s_M(\mathbb{A}_C)$, we have that

$$\sum_{j\in J(\mathbb{A}_C)} \tilde{r}_{s,j}\tilde{p}_s A_{s,j} \sum_{h\in H} \tilde{\theta}^h_j - \sum_{k\in\mathbb{A}_C} \sum_{h\in H} \tilde{\delta}^h_{s,k} = 0, \;\; \forall s \in S$$

and, as a particular case, $(\tilde{r}_{s,j})_{j\in J(\mathbb{A}_C)}$ satisfies the above equation. Consequently, whenever the primitives are not negotiated, the property (6.2) holds. Moreover, since $\sum_{h\in H} \tilde{\theta}^h_j = 0$ for all $m \in \{1, 2, \dots, n(\mathbb{A}_C)$, it follows from (19) that $\tilde{r}_{s,j^m(\mathbb{A}_C)} = \beta^{s,m}_M(\mathbb{A}_C)$ for all $j \in J(\mathbb{A}_C)$. Thus, $(\tilde{r}_{s,j^m(\mathbb{A}_C)})_{m=1}^{n(\mathbb{A}_C)}$ belongs to $\mathcal{R}^s_M(\mathbb{A}_C)$, and item (6.1) holds whenever the primitives are not negotiated.

Case II: *Suppose that* $\sum_{h\in H} \tilde{\varphi}^h_k > 0$ *for some* $k \in \mathbb{A}_C$.

It follows from (13) that $\sum_{h\in H} \tilde{\theta}^h_j > 0$ for all $j \in J(\mathbb{A}_C)$. Since $(\tilde{\pi}, \tilde{\eta})$ is a M-semi-equilibrium, we know that

$$\tilde{r}_{s,j^m(\mathbb{A}_C)} = \arg \max_{r\in[\beta^{s,m}_M(\mathbb{A}_C),1]} -\left(r\, F^{s,m}_{\mathbb{A}_C}(\tilde{\pi}, \tilde{\eta}) + \sum_{i=1}^{m-1} \tilde{r}_{s,j^i(\mathbb{A}_C)} F^{s,i}_{\mathbb{A}_C}(\tilde{\pi}, \tilde{\eta}) - \sum_{k\in\mathbb{A}_C} \sum_{h\in H} \tilde{\delta}^h_{s,k} \right)^2, \tag{20}$$

where $F^{s,i}_{\mathbb{A}_C}(\tilde{\pi}, \tilde{\eta}) := \tilde{p}_s A_{s,j^i(\mathbb{A}_C)} \sum_{h\in H} \tilde{\theta}^h_{j^i(\mathbb{A}_C)}$.

Moreover, as $\sum_{h\in H} \tilde{\theta}^h_j > 0$ for all $j \in J(\mathbb{A}_C)$, we have that $F^{s,i}_{\mathbb{A}_C}(\tilde{\pi}, \tilde{\eta}) = 0$ if and only if $\|A_{s,j^i(\mathbb{A}_C)}\|_1 = 0$. Now, define for each state of nature $s \in S$ the set $I^s_{\mathbb{A}_C} = \{m : \|A_{s,j^m(\mathbb{A}_C)}\|_1 \neq 0\}$.

If $I^s_{\mathbb{A}_C}$ is empty, then it follows from (19) that $\tilde{r}_{s,j^m(\mathbb{A}_C)} = 1$, for all $m \in \{1, 2, \dots, n(\mathbb{A}_C)\}$. Thus, item (6.1) holds in this case. Otherwise, suppose that $I^s_{\mathbb{A}_C} \neq \emptyset$ and consider the following claims,

Claim 1. *Given* $m \in I^s_{\mathbb{A}_C}$, *if*

$$F^{s,m}_{\mathbb{A}_C}(\tilde{\pi}, \tilde{\eta}) + \sum_{i=1}^{m-1} \tilde{r}_{s,j^i(\mathbb{A}_C)} F^{s,i}_{\mathbb{A}_C}(\tilde{\pi}, \tilde{\eta}) \leq \sum_{k\in\mathbb{A}_C} \sum_{h\in H} \tilde{\delta}^h_{s,k} \tag{21}$$

holds, then $\tilde{r}_{s,j^m(\mathbb{A}_C)} = 1$, *and* $\tilde{r}_{s,j^{m'}(\mathbb{A}_C)} = 1$ *for each* $m' < m$ *with* $m' \in I^s_{\mathbb{A}_C}$.

Proof: As $m \in I^s_{\mathbb{A}_C}$, if (21) holds, then $r_{s,j^m(\mathbb{A}_C)} = 1$ is the unique maximizer of the objective function in (20), and consequently $\tilde{r}_{s,j^m(\mathbb{A}_C)} = 1$. Now, suppose that there exists m' in $I^s_{\mathbb{A}_C}$ such that $\tilde{r}_{s,j^{m'}(\mathbb{A}_C)} < 1$ and $m' < m$. Since (21) holds for m,

$$\sum_{i=1}^{m'} \tilde{r}_{s,j^i(\mathbb{A}_C)} F^{s,i}_{\mathbb{A}_C}(\tilde{\pi}, \tilde{\eta}) < \sum_{k\in\mathbb{A}_C} \sum_{h\in H} \tilde{\delta}^h_{s,k}, \tag{22}$$

which is a contradiction with auctioneer optimality condition (3.6). Therefore, $\tilde{r}_{s,j^{m'}(\mathbb{A}_C)} = 1$. ☒

Claim 2. *Given* $m \in I^s_{\mathbb{A}_C}$, *if*

$$\beta^{s,m}_M(\mathbb{A}_C) F^{s,m}_{\mathbb{A}_C}(\tilde{\pi}, \tilde{\eta}) + \sum_{i=1}^{m-1} \tilde{r}_{s,j^i(\mathbb{A}_C)} F^{s,i}_{\mathbb{A}_C}(\tilde{\pi}, \tilde{\eta}) < \sum_{k\in\mathbb{A}_C} \sum_{h\in H} \tilde{\delta}^h_{s,k} < F^{s,m}_{\mathbb{A}_C}(\tilde{\pi}, \tilde{\eta}) + \sum_{i=1}^{m-1} \tilde{r}_{s,j^i(\mathbb{A}_C)} F^{s,i}_{\mathbb{A}_C}(\tilde{\pi}, \tilde{\eta}) \tag{23}$$

holds, then $\tilde{r}_{s,j^m(\mathbb{A}_C)} \in (\beta^{s,m}_M(\mathbb{A}_C), 1)$,

$$\tilde{r}_{s,j^m(\mathbb{A}_C)} F^{s,m}_{\mathbb{A}_C}(\tilde{\pi}, \tilde{\eta}) + \sum_{i=1}^{m-1} \tilde{r}_{s,j^i(\mathbb{A}_C)} F^{s,i}_{\mathbb{A}_C}(\tilde{\pi}, \tilde{\eta}) = \sum_{k\in\mathbb{A}_C} \sum_{h\in H} \tilde{\delta}^h_{s,k}, \tag{24}$$

and $\tilde{r}_{s,j^{m'}(\mathbb{A}_C)} = 1$ *for each* $m' < m$ *with* $m' \in I^s_{\mathbb{A}_C}$.

PROOF: If (23) is satisfied, the global maximum of the objective function (20) is attainable and, therefore, (24) holds. Moreover, we have that $\tilde{r}_{s,j^m(\mathbb{A}_C)} \in (\beta_M^{s,m}(\mathbb{A}_C), 1)$. Now, suppose that there exists $m' < m$ such that $\tilde{r}_{s,j^{m'}(\mathbb{A}_C)} < 1$ and $m' \in I^s_{\mathbb{A}_C}$. Since (23) holds for m, we have that

$$\sum_{i=1}^{m'} \tilde{r}_{s,j^i(\mathbb{A}_C)} F^{s,i}_{\mathbb{A}_C}(\tilde{\pi},\tilde{\eta}) < \sum_{k\in\mathbb{A}_C}\sum_{h\in H} \tilde{\delta}^h_{s,k}, \tag{25}$$

which is a contradiction with (3.6). Therefore, $\tilde{r}_{s,j^{m'}(\mathbb{A}_C)} = 1$. ⊠

CLAIM 3. *Given $m \in I^s_{\mathbb{A}_C}$, if*

$$\beta_M^{s,m}(\mathbb{A}_C) F^{s,m}_{\mathbb{A}_C}(\tilde{\pi},\tilde{\eta}) + \sum_{i=1}^{m-1} \tilde{r}_{s,j^i(\mathbb{A}_C)} F^{s,i}_{\mathbb{A}_C}(\tilde{\pi},\tilde{\eta}) \geq \sum_{k\in\mathbb{A}_C}\sum_{h\in H} \tilde{\delta}^h_{s,k}, \tag{26}$$

holds, then $\tilde{r}_{s,j^m(\mathbb{A}_C)} = \beta_M^{s,m}(\mathbb{A}_C)$ and $\tilde{r}_{s,j^{m'}(\mathbb{A}_C)} = \beta_M^{s,m'}(\mathbb{A}_C)$ for each $m' > m$ with $m' \in I(\mathbb{A}_C)$.

PROOF: If (26) holds, then $r_{s,j^m(\mathbb{A}_C)} = \beta_M^{s,m}(\mathbb{A}_C)$ is the unique maximizer of objective function in (20) and, therefore, $\tilde{r}_{s,j^m(\mathbb{A}_C)} = \beta_M^{s,m}(\mathbb{A}_C)$. Moreover, since (26) is valid for each $m' > m$, if $m' \in I(\mathbb{A}_C)$, then $\tilde{r}_{s,j^{m'}(\mathbb{A}_C)} = \beta_M^{s,m}(\mathbb{A}_C)$. ⊠

Now, we can easily see that each $m \in I^s_{\mathbb{A}_C}$ satisfies the conditions of one and only one Claim. Additionally, the set of $m \in I^s_{\mathbb{A}_C}$ that satisfies the conditions of a specific Claim may be empty. Moreover, the following facts are valid:

- There exists at most one m for which conditions of Claim 2 holds.
- If $m \in I^s_{\mathbb{A}_C}$ satisfies the condition of Claim 1 or 2, then each $m' < m$, with $m' \in I^s_{\mathbb{A}_C}$, satisfies the condition of Claim 1.
- If $m \in I^s_{\mathbb{A}_C}$ satisfies the condition of Claim 2 or 3, then each $m'' > m$, with $m'' \in I^s_{\mathbb{A}_C}$, satisfies the condition of Claim 3.

Therefore, suppose that there exists $m \in I^s_{\mathbb{A}_C}$ that satisfies the condition of Claim 2. Then, it follows from items above and (19) that (i) $\tilde{r}_{s,j^{m'}(\mathbb{A}_C)} = 1$, for all $m' < m$; (ii) $\tilde{r}_{s,j^m(\mathbb{A}_C)} \in (\beta_M^{s,m}(\mathbb{A}_C), 1)$; and (iii) $\tilde{r}_{s,j^{m''}(\mathbb{A}_C)} = \beta_M^{s,m''}(\mathbb{A}_C)$, for all $m'' > m$. This guarantees that condition (6.1) holds in this case.

If there exists no $m \in I^s_{\mathbb{A}_C}$ that satisfies condition of Claim 2, we have two possibilities:

- There exists $m \in I^s_{\mathbb{A}_C}$ such that $\tilde{r}_{s,j^m(\mathbb{A}_C)} = \beta_M^{s,m}(\mathbb{A}_C)$. In this case, define $\tilde{m} = \min\{m' \in I^s_{\mathbb{A}_C} : \tilde{r}_{s,j^{m'}(\mathbb{A}_C)} = \beta_M^{s,m'}(\mathbb{A}_C)\}$. Items above guarantee that $\tilde{m}$ satisfies the condition of Claim 3. This implies, using (19), that

$$\tilde{r}_{s,j^{m'}(\mathbb{A}_C)} = 1, \quad \forall\, m' < \tilde{m};$$

$$\tilde{r}_{s,j^{m'}(\mathbb{A}_C)} = \beta_M^{s,m'}(\mathbb{A}_C), \quad \forall\, m' > \tilde{m}.$$

- All $m \in I^s_{\mathbb{A}_C}$ satisfy $\tilde{r}_{s,j^m(\mathbb{A}_C)} = 1$. Thus, $\tilde{r}_{s,j^{m'}(\mathbb{A}_C)} = 1$, for all $m' \in \{1,2,\dots,n(\mathbb{A}_C)\}$.

Therefore, condition (6.1) always holds.

We will now prove that (6.2) always holds when $\sum_{h\in H}\tilde{\theta}^h_j > 0$ for all $j \in J(\mathbb{A}_C)$.

Note that, analogously to equation (15), it follows from equations (13) and (14), jointly with Assumption 5 that

$$\sum_{k\in\mathbb{A}_C}\sum_{h\in H}\tilde{\delta}^h_{s,k} = \sum_{\{k\in\mathbb{A}_C:\,\tilde{q}_k\neq 0\}}\sum_{h\in H}\tilde{\delta}^h_{s,k} \leq \sum_{\{k\in\mathbb{A}_C:\,\tilde{q}_k\neq 0\}}\tilde{p}_s A_{s,k}\sum_{h\in H}\tilde{\varphi}^h_k = \tilde{p}_s\sum_{\{k\in\mathbb{A}_C:\,\tilde{q}_k\neq 0\}}A_{s,k}\max_{j\in J(\mathbb{A}_C)}\sum_{h\in H}\tilde{\theta}^h_j \leq \tilde{p}_s\sum_{j\in J(\mathbb{A}_C)}A_{s,j}\sum_{h\in H}\tilde{\theta}^h_j. \tag{27}$$

Thus, it follows from (3.6) and (27) that having $\sum_{j\in J(\mathbb{A}_C)}\tilde{r}_{s,j}\tilde{p}_s A_{s,j}\sum_{h\in H}\tilde{\theta}^h_j < \sum_{k\in\mathbb{A}_C}\sum_{h\in H}\tilde{\delta}^h_{s,k}$, would lead us to a contradiction. Therefore, $\sum_{j\in J(\mathbb{A}_C)}\tilde{r}_{s,j}\tilde{p}_s A_{s,j}\sum_{h\in H}\tilde{\theta}^h_j \geq \sum_{k\in\mathbb{A}_C}\sum_{h\in H}\tilde{\delta}^h_{s,k}$.

Now, suppose that $\min_{k\in\mathbb{A}_C}\{||A_{s,k}||_1,\,\overline{c}_{s,k}\} > 0$ and define $m^* = \min\{m' : \|A_{s,j^{m'}(\mathbb{A}_C)}\|_1 \neq 0\}$. Thus, it follows from the definition of $\Upsilon^s_M(\mathbb{A}_C)$ that $\beta^{s,m}_M(\mathbb{A}_C) = \frac{1}{M_1}$ for all $m \leq m^*$. Analogously to the argument made in Case II of the proof of Lemma 5, we have that $\min_{k\in\mathbb{A}_C}\{||A_{s,k}||_1,\,\overline{c}_{s,k}\} > 0$ if and only if $\min_{k\in\mathbb{A}_C}\{||A_{s,k}||_1,\,\underline{c}_{s,k}\} > 0$, and that

$$\sum_{k\in\mathbb{A}_C}\sum_{h\in H}\tilde{\delta}^h_{s,k} \geq \underline{p}\min_{k\in\mathbb{A}_C}\left\{\|A_{s,k}\|_1,\underline{c}_{s,k}\right\}\max_{j\in J(\mathbb{A}_C)}\sum_{h\in H}\tilde{\theta}^h_j.$$

Therefore, we know that $\varsigma^s(\mathbb{A}_C) := \underline{p}\,\min_{k\in\mathbb{A}_C}\left\{\|A_{s,k}\|_1,\underline{c}_{s,k}\right\}$ is strictly positive and that $\|A_{s,j^{m^*}(\mathbb{A}_C)}\|_1 > 0$. Thus, there exists $M_1^\star(\mathbb{A}_C) > M_1^{\star\star\star}$ such that for each M-semi-equilibrium, with $M_1 > M_1^\star(\mathbb{A}_C)$, we have that $\frac{1}{M_1}\|A_{s,j^{m^*}(\mathbb{A}_C)}\|_1 < \varsigma^s(\mathbb{A}_C)$. Then,

$$\frac{1}{M_1}\tilde{p}_s A_{s,j^{m^*}(\mathbb{A}_C)}\sum_{h\in H}\tilde{\theta}^h_{j^{m^*}(\mathbb{A}_C)} < \varsigma^s(\mathbb{A}_C)\sum_{h\in H}\tilde{\theta}^h_{j^{m^*}(\mathbb{A}_C)} \leq \sum_{k\in\mathbb{A}_C}\sum_{h\in H}\tilde{\delta}^h_{s,k}, \tag{28}$$

and it follows from the fact that, for any $m < m^*$, $\|A_{s,j^m(\mathbb{A}_C)}\|_1 = 0$ and from (3.6) that

$$\sum_{m=1}^{m^*}\tilde{r}_{s,j^m(\mathbb{A}_C)}\tilde{p}_s A_{s,j^m(\mathbb{A}_C)}\sum_{h\in H}\tilde{\theta}^h_{j^m(\mathbb{A}_C)} - \sum_{k\in\mathbb{A}_C}\sum_{h\in H}\tilde{\delta}^h_{s,k} \leq 0. \tag{29}$$

Thus, when $\min_{k\in\mathbb{A}_C}\{||A_{s,k}||_1,\,\overline{c}_{s,k}\} > 0$, if $\tilde{r}_{s,j^m(\mathbb{A}_C)} = \beta^{s,m}_M(\mathbb{A}_C)$, we have that $\tilde{r}_{s,j^m(\mathbb{A}_C)} = \frac{1}{M_2}$, because otherwise $\tilde{r}_{s,j^m(\mathbb{A}_C)} = \frac{1}{M_1}$, which implies that $m \leq m^\star$, contradicting (28) and (3.6).

Furthermore, when $\min_{k\in\mathbb{A}_C}\{||A_{s,k}||_1,\,\overline{c}_{s,k}\} = 0$, from definition we have that $\beta^{s,m}_M(\mathbb{A}_C) = \frac{1}{M_2}$ for each $m \in \{1,2,\ldots,n(\mathbb{A}_C)\}$ and, consequently, if $\tilde{r}_{s,j^m(\mathbb{A}_C)} = \beta^{s,m}_M(\mathbb{A}_C)$, then $\tilde{r}_{s,j^m(\mathbb{A}_C)} = \frac{1}{M_2}$.

Now, it follows from Claims above that $\sum_{i=1}^m\tilde{r}_{s,j^i(\mathbb{A}_C)}\tilde{p}_s A_{s,j^i(\mathbb{A}_C)}\sum_{h\in H}\tilde{\theta}^h_{j^i(\mathbb{A}_C)} - \sum_{k\in\mathbb{A}_C}\sum_{h\in H}\tilde{\delta}^h_{s,k}$ is greater than zero if and only if $\tilde{r}_{s,j^m(\mathbb{A}_C)} = \beta^{s,m}_M(\mathbb{A}_C) = \frac{1}{M_2}$. Thus, define $m^{**} = \min\{m : \tilde{r}_{s,j^m(\mathbb{A}_C)} = \frac{1}{M_2}\}$. Note that $m^* < m^{**}$.

Finally,

$$\begin{aligned}
0 &\leq \sum_{j\in J(\mathbb{A}_C)}\tilde{r}_{s,j}\tilde{p}_s A_{s,j}\sum_{h\in H}\tilde{\theta}^h_j - \sum_{k\in\mathbb{A}_P}\sum_{h\in H}\tilde{\delta}^h_{s,k}\\
&= \sum_{m=1}^{m^{**}-1}\tilde{r}_{s,j^m(\mathbb{A}_C)}F^{s,m}_{\mathbb{A}_C}(\tilde{\pi},\tilde{\eta}) + \sum_{m=m^{**}}^{n(\mathbb{A}_C)}\tilde{r}_{s,j^m(\mathbb{A}_C)}F^{s,m}_{\mathbb{A}_C}(\tilde{\pi},\tilde{\eta}) - \sum_{k\in\mathbb{A}_C}\sum_{h\in H}\tilde{\delta}^h_{s,k}\\
&\leq \sum_{m=m^{**}}^{n(\mathbb{A}_C)}\frac{1}{M_2}\tilde{p}_s A_{s,j^m(\mathbb{A}_C)}\sum_{h\in H}\tilde{\theta}^h_{j^m(\mathbb{A}_C)}\\
&\leq \frac{2}{M_2}\sum_{j\in J(\mathbb{A}_C)}\|A_{s,j}\|_1(\#H)^2\,\Omega,
\end{aligned}$$

which guarantees that item (6.2) always holds.

Therefore, lemma holds taking $M_1^\star = \max_{\mathbb{A}_C \subset \mathcal{A}_C} M_1^\star(\mathbb{A}_C)$. □

LEMMA 7. *For each $M = (M_1, M_2) \in \mathcal{M}$ with $M_1 > M_1^\star$, there exists a M-semi-equilibrium $(\tilde{\pi}, \tilde{\eta})$ in which conditions (5.1), (5.2), (6.1) and (6.2) hold and the following properties are satisfied:*

(7.1) *For each $s \in S$ and $l \in L$,*

$$\sum_{h\in H} \tilde{x}^h_{s,l} - (Y_s W_0)_l - W_{s,l} \leq \frac{2}{M_2} \sum_{i\in\{P,C\}} \sum_{\mathbb{A}_i \subset \mathcal{A}_i} \sum_{j\in J(\mathbb{A}_i)} \|A_{s,j}\|_1 (\#H)^2\,\Omega;$$

(7.2) *For each $h \in H$, $\hat{\Psi}^h(\tilde{p}_{-0}, \tilde{\eta}^h) \cap B^h(\tilde{\pi}) = \emptyset$.*

PROOF: We know from Lemma 6 that there exists, for each $M \in \mathcal{M}$ with $M_1 > M_1^\star$, a M-semi-equilibrium $(\tilde{\pi}, \tilde{\eta})$ that satisfies conditions (5.1), (5.2), (6.1) and (6.2). Therefore, fix $(\tilde{\pi}, \tilde{\eta})$ in which all the above properties hold. Item (7.1) follows directly from items (3.8), (5.2) and (6.2).

We already know from item (3.2) that $\hat{\Psi}^{h,M}(\tilde{p}_{-0}, \tilde{\eta}^h) \cap B^h_M(\tilde{\pi}) = \emptyset$. Suppose that it exists $y \in \hat{\Psi}^h(\tilde{p}_{-0}, \tilde{\eta}^h) \cap B^h(\tilde{\pi})$. It follows from the definition of augmented preferences that for $\lambda \in (0,1]$ sufficiently small, $z := \lambda y + (1-\lambda)\tilde{\eta}^h \in \hat{\Psi}^h(\tilde{p}_{-0}, \tilde{\eta}^h)$ and $\|z\|_\infty < M_1$, because $\|\tilde{\eta}^h\|_\infty < M_1$. Therefore, as $z \in B^h_M(\tilde{\pi})$ we have a contradiction with $\hat{\Psi}^{h,M}(\tilde{p}_{-0}, \tilde{\eta}^h) \cap B^h_M(\hat{\pi}) = \emptyset$. This concludes the proof of item (7.2). □

Finally, the proof of Theorem 1 is a direct consequence of Lemma below.

LEMMA 8. *There exists a non-trivial equilibrium for the economy $\mathcal{E}(S^*, \mathcal{H}, \mathcal{L}, \mathcal{F})$, which can be obtained as the limit of a sequence of M-semi-equilibriums when M_2 goes to infinity and $M_1 > M_1^\star$.*

PROOF: We know from Lemma 7 that there exists, for each $M \in \mathcal{M}$ with $M_1 > M_1^\star$, a M-semi-equilibrium $(\tilde{\pi}_M, \tilde{\eta}_M)$ that satisfies conditions (5.1), (5.2), (6.1), (6.2), (7.1) and (7.2). Thus, fix a $M_1 > M_1^\star$ and construct a sequence of M-semi-equilibriums $(\tilde{\pi}_{M_2}, \tilde{\eta}_{M_2})$, indexed only by M_2, which satisfy the above conditions for all M_2. It follows from the fact that $(\tilde{\pi}_{M_2}, \tilde{\eta}_{M_2})$ belongs to a compact set, independent of M_2, that there exists a convergent subsequence. We will denote the limit of this subsequence as $(\hat{\pi}, \hat{\eta})$.

It is straightforward that items (3.3), (3.4) and (5.1) still hold for the limit allocation $(\hat{\pi}, \hat{\eta})$. Moreover, one can easily see that at the limit items (3.1), (5.2), (6.2) and (7.1) become, respectively,

(3.1*) For each $h \in H$, $\hat{\eta} \in B^h(\hat{\pi})$;

(5.2*) For each $\mathbb{A}_P \in \mathcal{A}_P$ and each $s \in S$,

$$\sum_{j\in J(\mathbb{A}_P)} \hat{r}_{s,j}\hat{p}_s A_{s,j} \sum_{h\in H} \hat{\theta}^h_j - \sum_{k\in\mathbb{A}_P} \sum_{h\in H} \hat{\delta}^h_{s,k} = 0;$$

(6.2*) For each $\mathbb{A}_C \in \mathcal{A}_C$ and each $s \in S$,

$$\sum_{j\in J(\mathbb{A}_C)} \hat{r}_{s,j}\hat{p}_s A_{s,j} \sum_{h\in H} \hat{\theta}^h_j - \sum_{k\in\mathbb{A}_C} \sum_{h\in H} \hat{\delta}^h_{s,k} = 0;$$

(7.1*) For each $s \in S$ and $l \in L$,

$$\sum_{h\in H} \hat{x}^h_{s,l} - (Y_s W_0)_l - W_{s,l} \leq 0;$$

where item (3.1*) follows from the closed graph of the budget set correspondence B^h.

Moreover, we know that, for each M_2, the second-periods budget constraints are satisfied with equality. Then, the limit second period budget constraints still hold with equality. This fact, jointly with the items (5.2*), (6.2*) and (7,1*) above, imply that, for each $(s,l) \in S \times L$,

$$\sum_{h \in H} \hat{x}^h_{s,l} - (Y_s W_0)_l - W_{s,l} = 0. \tag{30}$$

Note that, every convergent sequence of elements belonging to $\mathcal{R}^s_M(\mathbb{A}_C)$ for each M_2 has a limit at $\mathcal{R}(\mathbb{A}_C)$. This implies that $(\hat{r}_{s,j})_{j \in J(\mathbb{A}_C)} \in \mathcal{R}(\mathbb{A}_C)$ for all $\mathbb{A}_C \subset \mathcal{A}_C$ and $s \in S$. Moreover, it follows from the fact that M_1 is fixed and from the definition of $\mathcal{R}^s_M(\mathbb{A}_C)$ that if $\min_{k \in \mathbb{A}_C} \{||A_{s,k}||_1, \bar{c}_{s,k}\} > 0$, for a given class of primitives $\mathbb{A}_C \subset \mathcal{A}_C$ and state $s \in S$, we have that $(\tilde{r}_{s,j^m_{\mathbb{A}_C}})_{M_2} \geq \frac{1}{M_1}$ for all $m \leq m^\star$ and for all M_2, where $m^\star := \min\{m : \|A_{s,j^m(\mathbb{A}_C)}\|_1 \neq 0\}$. Analogously, if $\min_{k \in \mathbb{A}_P} \{||A_{s,k}||_1, \bar{c}_{s,k}\} > 0$, for a given class of primitives $\mathbb{A}_P \subset \mathcal{A}_P$ and state $s \in S$, we have that $(\tilde{r}_{s,\mathbb{A}_P})_{M_2} \geq \frac{1}{M_1}$

Therefore, the limit expected rates of payment satisfy

$$\left[\min_{k \in \mathbb{A}_C} \{||A_{s,k}||_1, \bar{c}_{s,k}\} > 0\right] \Rightarrow \hat{r}_{s,j^m_{\mathbb{A}_C}} \geq \frac{1}{M_1}, \; \forall m \leq m^\star$$

and

$$\left[\min_{k \in \mathbb{A}_P} \{||A_{s,k}||_1, \bar{c}_{s,k}\} > 0\right] \Rightarrow \hat{r}_{s,\mathbb{A}_P} > \frac{1}{M_1},$$

which implies, using the fact that $\hat{p}_{s,l} \geq \underline{p}$, for all (s,l), that

$$[\min\{\hat{p}_s Y_s C_k(\hat{p}_0, \hat{q}_k); \hat{p}_s A_{s,k}\} > 0, \;\; \forall k \in \mathbb{A}_C] \Rightarrow \hat{r}_{s,j^m_{\mathbb{A}_C}} > 0, \; \forall m \leq m^\star,$$

and

$$[\min\{\hat{p}_s Y_s C_k(\hat{p}_0, \hat{q}_k); \hat{p}_s A_{s,k}\} > 0, \;\; \forall k \in \mathbb{A}_P] \Rightarrow \hat{r}_{s,\mathbb{A}_P} > 0.$$

In order to prove the optimality of the limit $(\hat{\pi}, \hat{\eta})$, we will first show that, for a given agent $h \in H$, there is nothing in the interior of the budget set that is strictly preferred that $\hat{\eta}^h$. Suppose that there is an allocation y such that $y \in \hat{\Psi}^h(\hat{p}_{-0}, \hat{\eta}^h) \cap \dot{B}^h(\hat{\pi})$. Since $\hat{\Psi}^h$ is lower hemicontinuous and $(\tilde{\pi}_{M_2}, \tilde{\eta}_{M_2}) \to (\hat{\pi}, \hat{\eta})$, there exists $y_{M_2} \in \hat{\Psi}^h((\tilde{p}_{-0})_{M_2}, \tilde{\eta}_{M_2})$ such that $y_{M_2} \to y$. Since $\dot{B}_M$ has open values, for M_2 sufficiently large, $y_{M_2} \in \hat{\Psi}^h((\tilde{p}_{-0})_{M_2}, \tilde{\eta}^h_{M_2}) \cap \dot{B}^h(\tilde{\pi}_{M_2})$, which is a contradiction with (7.2). Thus, $\hat{\Psi}^h(\hat{p}_{-0}, \hat{\eta}^h) \cap \dot{B}^h(\hat{\pi}) = \emptyset$. Moreover, it follows from Lemma 4 that the M-semi-equilibrium commodity prices are lower bounded and, therefore, the limit allocation has prices strictly greater than zero. This implies that the interior of the limit budget set has non-empty values, $\dot{B}^h(\hat{\pi}) \neq \emptyset$. Now, as B^h also has convex values, we have that the closure of $\dot{B}^h(\hat{\pi})$ is equal to the original budget set, $B^h(\hat{\pi})$. Then, it follows that $\hat{\Psi}^h(\hat{p}_{-0}, \hat{\eta}^h) \cap B^h(\hat{\pi}) = \emptyset$. Since $\Psi^h(\hat{p}_{-0}, \hat{\eta}^h) \subset \hat{\Psi}^h(\hat{p}_{-0}, \hat{\eta}^h)$, we have that $\Psi^h(\hat{p}_{-0}, \hat{\eta}^h) \cap B^h(\hat{\pi}) = \emptyset$. That completes the proof of optimality.

Finally, given a class of primitives $\mathbb{A}_i$, for which there exists at least one derivative $j \in J(\mathbb{A}_i)$ that have positive rates of payment (in at least one state of nature), optimality conditions on agents allocations guarantee that the price $\hat{q}_j > 0$. Thus, as $(\hat{p}, \hat{q}_K, \hat{q}_J) \in \Xi$, there exists at least one primitive $k \in \mathbb{A}_i$ for which $\hat{q}_k > 0$. This concludes the proof. □

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