TEXTO PARA DISCUSSÃO

No. 520

A Relational Theory of Relationship Lending Under Contractual Incompleteness

Vinicius Carrasco
João M P de Mello

DEPARTAMENTO DE ECONOMIA
www.econ.puc-rio.br

Repeated Lending under Contractual Incompleteness*

Vinicius Carrasco and João M P De Mello†

December 2008

Abstract

We consider a model of repeated (relationship) lending in which some contingencies that are relevant for a bank's decision to finance a project cannot be described contractually. The hazards related to this lack of contractibility can be magnified by actions taken by an entrepreneur. The continuation value of a lending relationship induces borrowers to take actions that minimize the *ex-post* conflict of interests resulting from contractual incompleteness. The optimal lending relationship is stationary on the equilibrium path. A robust feature of an optimal lending relationship is that the action schedule (as a function of project types) adopted by the entrepreneur is either a constant or a step function. Hence, the bank imposes to the entrepreneur a finite set of decisions from which he can pick his action, bounding his discretion over decisions. This leads to lower interest rates charged by the bank and to efficient refinancing in a lending relationship when compared to arm's length financing.

KEYWORDS: Relational Lending, Contractual Incompleteness, Repeated Games with Asymmetric Information.

JEL CODES: C73, G21, D82.

1 Introduction

Repeated lender-borrower interaction, often shorthanded as Relational Lending, is viewed as a major technology to produce loans. Indeed, empirical evidence suggests that relationships are valuable for different types of borrowers. Berger and Udell [1995] and Petersen and Rajan [1994] document that small firms with established banking relationships have more and better access to bank credit. Close bank-firm ties are pervasive in several financial systems such as the Japanese Main Bank and the German Universal Bank systems (Baums [1995]; Aoki at el [1995]). Hoshi, Kashyap and Scharfstein [1991] document that large firms with close ties to banks face less difficulty in raising capital than those without banking relationships.

Besides the empirical evidence, a full understanding of relational lending has become particularly important in light of the recent subprime crisis. Although the academic literature has yet to produce a full account of events, excessive disintermediation is one likely culprit. Splitting the roles of origination and lending may have lowered the quality of underwriting. Fragmentation of creditors reduce the incentives to discipline

*We would like to thank Antônio Carlos Sodré, and seminar participants at CEMFI's MADMAC for comments and suggestions. All errors are ours.

†Carrasco: Departamento de Economia, PUC-Rio, vnc@econ.puc-rio.br. De Mello: Departamento de Economia, PUC-Rio, jmpm@econ.puc-rio.br.

borrowers. The US banking system is already mutating towards an Universal Banking structure where investment banking will lose importance. This is a movement towards more intermediation and, consequently, ever increasing importance of relationships as lending technologies.

Our contribution is to develop a theory of repeated lending in which relationships play a disciplinary role. In the model, a wealth constrained entrepreneur has access to a project that, on top of some initial funding, may need, with some probability, additional resources to go through. If these additional funds are not provided, the project is shut down. Three interacting features make the provision of additional funds non-trivial for the bank. First, the entrepreneur is privately informed about the probability of the project going through *without* the need of more funds. Second, upon the project being granted more funds, its return will depend on some *non-contractible* states. Third, and most important, the entrepreneur may take *actions* that affect the likelihood of his most preferred state in case he is granted more funds. Hence, contingencies that are relevant for a bank's decision to finance a project cannot be described contractually, and the hazards related to the lack of contractibility may be magnified by actions taken by an entrepreneur.

When the bank and the entrepreneur interact just once (arm's length financing), the bank mitigates the hazards related to contractual incompleteness by offering two different contracts to the entrepreneur. In the first contract, only the initial funds are provided. The second contract prescribes the provision of additional funds (a credit line), but the entrepreneur is charged an up-front premium for these funds. One interpretation is that, given the lack of contractibility of some states, the provision of additional funds to the entrepreneur grants him *control* over relevant decisions. Anticipating that the entrepreneur will take actions that goes against its interests, the bank charges up-front for such a delegation of control. Given these two contracts, different "types" of entrepreneurs will sort themselves. If the likelihood of the project going through is high, the entrepreneur will pick the first contract. Therefore, projects that would be refinanced in a world of contractual completeness will be shut down with positive probability under arm's length financing.

In a lending relationship, the bank and the entrepreneur interact repeatedly, and the former may condition future play on the latter's current behavior. The continuation values of a relationship induce borrowers to take actions that are more favorable to the lender. This alignment of parties' interests reduces the inefficiencies related to the lack of contractibility.[1]

In spite of the complexity of a setting in which the players' interact repeatedly and one of them (the borrower) has private information and takes non-contractible actions, we fully characterize the optimal lending relationship. In fact, the optimal lending relationship takes a surprisingly simple form. We first show that, along the equilibrium path, the optimal lending relationship is stationary: at each period and for every history of events that happens with positive probability in equilibrium, the entrepreneur is promised the highest possible continuation value. In other words, for any two actions, a_1 and a_2, taken with positive

[1] The repeated dimension of relationship is a substitute to the "hands-on" monitoring done, for example, by Japanese banks. In the banking literature, relationships are considered a technology to produce three types of monitoring: *ex-ante*, *interim* and *ex-post* (Aoki and Dinç [2000]). *Ex-ante* monitoring, or screening, solves hidden information (adverse selection) problems. By interacting repeatedly with borrowers, lenders learn relevant borrower (persistent) private information. *Interim* monitoring refers to the ability that relational banks may have in influencing borrowers' decisions in the making. By holding equity and control stake, Japanese and German bankers are able to affect decisions directly (Baums [1995]). In the small firm context, privileged access to detailed transactions account information allows banks that interact repeatedly with borrowers to better access their financial status before things get out of control (Mester, Nakamura and Renault [2004]). Finally, *ex-post* monitoring refers to verification after the fact, i.e., after the relevant decision have been made. Most *ex-post* monitoring occurs in situations of financial distress, when the net worth of the firm (or the project) is not verifiable.

likelihood[2], the entrepreneur will be promised the *same* continuation value, $v(a_1) = v(a_2)$. Off-equilibrium, i.e. whenever the entrepreneur takes actions that are *not* prescribed to be taken, he is promised the lowest continuation value, which is the value associated with the indefinite repetition of arm's length contracting. Such punishment guarantees that the entrepreneur will stick the course of action implicitly agreed by the relationship.

Our second result is that all projects are refinanced if the entrepreneur is patient. Along with the fact that on the equilibrium path the entrepreneur is promised the highest possible continuation values at each period, this implies that the implicit contract between the lender and the entrepreneur satisfies a strong form of renegotiation-proofness: current (as all projects are refinanced) as well as future (as only the highest continuation value is promised to the entrepreneur) efficiency is attained.

As repetition aligns the bank's and the entrepreneur's incentives, it is not so surprising that refinancing is granted to all projects. What is surprising is the *form* by which the alignment of interests take. Indeed, we show that, at an optimal lending relationship, the action schedule (as a function of project types) adopted by the entrepreneur is either a constant or a step function. Therefore, the entrepreneur *refrains* (at least partly when the schedule is a step function) from using his private information in deciding which action to pursue in an optimal lending relationship. One can interpret this result as saying that the bank imposes to the entrepreneur a finite set of decisions from which he can pick his action. This bounds the entrepreneur's discretion over decisions, minimizing the ex-post conflicts of interest.

The reason why it is optimal to restrict the entrepreneur's discretion is straightforward. As argued above, the bank charges up-front whenever the entrepreneur exercises (and abuses) the control granted by the provision of funds. If he chooses actions that do not depend on the project types, the entrepreneur refrains from exercising (abusing) control. As the entrepreneur abstains from using his private information when taking actions (and also ends up picking actions that are closer to the bank's preferred one in a lending relationship), the repayment (premium) asked by the bank is smaller than under arm's length financing. Absent long-term engagements, borrower and lender interests are misaligned, and credit would be prohibitively expensive in situations in which refinancing is efficient.

In contrast to our paper, learning private information from borrowers is behind most rationales for relationships in the literature. By interacting repeatedly, lenders acquire (borrower's) private information about *ex-ante* identical borrowers, which is valuable to lend in subsequent periods. Sharpe [1990], Petersen and Rajan [1995], Rajan [1992] and Boot and Thakor [2000] all share this feature. Dynamics only play a role insofar as information is revealed over time. If all private information about borrowers could be acquired by lending once, relations would be maintained only if information is kept private to the relationship. Therefore, the explanation based on hidden information hinges on the emergence of an *ex-post* informational monopoly: relationships are valuable insofar as the acquired private information remains private to the initial lender (Aoki and Dinç [2000]).[3]

Much as in Boot and Thakor [2000], we show that competition does not harm relationship lending. In their paper, relationship loaning involves the bank adding value to a project through costly investment in specialization. The possibility of adding value differentiates the relational bank from other loaners, and is

[2] This means that there exist project types θ_1 and θ_2 for which the entrepreneur takes, respectively, actions a_1 and a_2 in equilibrium.

[3] Rajan [1992] explicitly models this trade-off, and relational lending (informed lending in his words) is costly because rent extraction by the bank dampens borrowers' incentive to exert effort.

a source of surplus to the bank which remains beneficial under competition. In our model, in contrast, the value generated by a relationship comes fully from its disciplining effect on the borrower. Competition among banks transfer the efficiency gains brought up by the relationship to the entrepreneur, which makes his implicit commitment to the relationship more credible.

There are some other papers that hints at disciplining explanations for relationship lending. In both Stiglitz and Weiss [1983] and Bolton and Scharfstein [1990], future lending prospects induce truth-telling on early period's profit realization and, consequently, assures that the financier recoups the funds provided. Our paper, however, is significantly different in terms of the contractual implications of a long-term relationship. In both papers, as in most "screening" models, an optimal long-term financing relationship calls for *interim* inefficiencies: whenever the borrower reports low profits, lenders "punish" by denying credit in the subsequent period. In our model, relationships arise exactly to overcome such inefficiencies, which are avoided through the endogenous commitment of the entrepreneur to not act against the bank.

In an infinitely repeated moral hazard game, Boot and Thakor [1994] study the intertemporal use of collateral in a competitive credit market. In their setting, the use of collateral is an inefficient way to induce an entrepreneur to exert effort that increases the likelihood of success of a project. Assuming lenders only need to break-even intertemporally (not every period), repetition and collateral become substitutes as mechanisms to align incentives: banks' ability to commit to a promise of better future contracts makes collateral less necessary to induce entrepreneurs' high effort. In our model, intertemporal use of collateral plays no role. Therefore, our theory also applies to empirical settings in which borrowers have little or no assets to set aside as collateral, such as the small firm credit market, or angel and venture capital financing. More importantly, in their paper, the lenders' ability to commit to better future contracts is crucial. In our set-up there is no need for long-term *explicit* contracts to be written. Also, in our model, banks are assumed to break-even period by period.

Our work also relates to (a more general) literature on repeated contracting. Athey et al [2005], for example, show that a privately informed monetary authority chooses, in certain states, a monetary policy that is unresponsive to the state of the economy. In our paper, actions are insensitive to private information for *all* possible parameter values and realizations of state. Full discretion – a static best response – while sometimes is optimal for a (positive probability) set of states in their model, is never optimal in ours. The force behind non-discretion in their paper – the log-concavity of the distribution of shocks – is also present in our model, but here there is an additional and more important force pushing towards ignoring private information. By reducing the probability of the entrepreneur's preferred (non-contractible) state, an action schedule that is invariant to the entrepreneur's private information reduces the interest rate on refinancing *irrespective* of distributional assumptions. This additional and new force suggests that it may be optimal for the entrepreneur to refrain from using its private information when choosing her actions even if the distribution of states is not log-concave.

Methodologically, we borrow from Abreu, Pearce and Stachetti (henceforth APS) [1990] the tools to solve for the set of Public Pure Strategy (PPS) Equilibria of the game played by a bank and an entrepreneur.

The paper is organized as follows. Section 2 describes the primitives of the model. Section 3 characterizes the equilibrium in the static game. Section 4 characterizes the best lending relationship. The analysis is split in two. In subsection 4.3, we consider the case in which the borrower is arbitrarily patient. In subsection 4.4, the case of an impatient borrower is considered. Section 5 concludes.

2 Set-Up: The Stage Game

We consider a setting in which a risk neutral entrepreneur has access to an indivisible project but lacks resources to fund it. There is a large number of risk neutral lenders, called banks, who have the means to finance the project, and compete among themselves to do so. The project needs $I_0 > 0$ to be started. If funds are provided, the project yields contractible returns $R > I_0$ with probability $\theta \in \left[\underline{\theta}, \overline{\theta}\right] \subset [0,1]$, where $\underline{\theta}$ is close to zero and $\overline{\theta}$ is close to 1. The entrepreneur is assumed to be privately informed about such probability. From the banks' perspective, θ is distributed according to a log-concave distribution $F(\theta)$, with density $f(\theta) > 0$.

With probability $1-\theta$, the project needs additional funding $I_1 > 0$ to go through. If I_1 is not provided, the project is shut down and yields zero in revenues. In case the funds are provided, the project succeeds with certainty but its exact return depends on the realization of a non-contractible state of nature. There are two possible states, E and F. In state E, the project yields R_E, while in state F it yields R_F.

We assume that both returns are not contractible, and that R_E fully accrues to the entrepreneur, while R_F fully accrues to the banks. Although extreme, this assumption captures in the clearest way the *ex-post* conflict of interests between the lender and the borrower arising from the lack of contractibility.

It is assumed throughout that

$$\min\left\{\left[1-E\left(\theta\right)\right]R_E, \left[1-E\left(\theta\right)\right]R_F\right\} > I_1,$$

so that refinancing is always efficient, and that R is large enough so that the entrepreneur is able to use the contractible return to "subsidize" any premium charged by the bank for the additional resources.[4]

The initial probability of state E is $\underline{a} \in (0,1)$. However, right after I_0 is provided, the entrepreneur can, at a cost $c(a)$ – with $c\left(\underline{a}\right) = c'\left(\underline{a}\right) = 0$, and $c'(1) = \infty$ –, take an action $a \in [\underline{a}, 1]$ that increases the probability of his most favorable state. Although a is observed by both the bank and the entrepreneur, it is assumed to be non-observable by a third party (e.g., a court).

Real world examples of such type of action abound: extracting some of the project's resources for private benefits, diverting resources from the project to other endeavors or to cover losses incurred somewhere else are but a couple examples. Another popular example is excessive (from the banks' perspective) risk-taking: since an entrepreneur is the residual claimant of "upsides" in a project (while banks are the residual claimants of "downsides"), he, against the bank's will, can take decisions that increase the variability of a project's return.

For simplicity, we normalize the initial probability of E to satisfy

$$\left(1-\underline{a}\right)E\left[1-\theta\right]R_F = I_1. \tag{A1}$$

This says that, if the status-quo actions is taken, refinancing can be provided by the bank with no additional premium. As will be clear shortly, none of the results to follow depend on such normalization.

The entrepreneur and the bank can contract on whether the initial funds, I_0, are provided, and on the payment, D, to be paid by the entrepreneur to the bank in case the project yields R. The parties can also contract on whether the financier will refinance the entrepreneur. This decision, however, can neither be

[4] We also implicitly assume that the bank who provides the initial funds have an specific (unmodeled) contribution to the project that makes the return R_F non-transferable to some other bank. This precludes the possibility of competion among the banks for just the "second" part of the project.

contingent on the states, nor on the *ex-post* returns due to their lack of verifiability. We model the refinancing decision by means of a variable $x \in \{0,1\}$, with $x = 1$ meaning that the additional funds will be granted. An interpretation for $x = 1$ is that a credit line is advanced by the bank to the entrepreneur. A contract is a pair $\{D, x\}$.

2.1 Timing of Events

The timing of the events of the stage game is as follows. At period zero, the banks post contracts of the form $\{D, x\}$. In period 1, the entrepreneur learns θ, and picks a contract $\{D, x\}$ from one of the banks.

The entrepreneur chooses which action to pursue in period 2. In period 3, if the project goes through, that is, return R realizes, a payment of D is made to the bank according to the contract signed in period 1. Otherwise, the relevant state is learned. The refinancing decision is given by what was agreed by the period 1 contract.

2.2 The Repeated Game

The entrepreneur and one particular bank can interact repeatedly over time, t, which runs from zero to infinity. At each period t, they play the stage game described above with the project's quality, θ_t, being drawn in an i.i.d. fashion from $F(.)$. Since there is a large number of identical banks, we keep assuming that banks will – given their correct expectations about the entrepreneur's type – exactly break-even in every period t.

We make this assumption because, if any, the gains generated by a relationship will be solely implied by its disciplinary effect on the borrower. Hence, those gains do not depend on the bank's identity, which suggests that competition among banks in period zero for the relational/implicit contract with the lender would drive their expected profits to zero. All the results in the paper go through if the bank is granted any constant level of per period profits.

3 Perfect Bayesian Equilibrium in the Static Game

As it is widely known since Rothschild and Stiglitz [1976], a competitive model in which firms compete for customers with private information does not admit a pooling equilibrium in general. Hence, a situation in which either all projects are given additional funds, or in which neither of projects are granted funds cannot prevail as part of an equilibrium in the stage game.[5]

The only possibility, therefore, is that two contracts are offered in equilibrium: one that prescribes refinancing and the other which does not. They take the form of $\{\overline{D}, 1\}$, and $\{\underline{D}, 0\}$, with $\overline{D} > \underline{D}$, where such repayments guarantee, given its (correct) expectations, zero profits for the bank in each contract:

$$\overline{D}E(\theta|x=1) + E\left[(1-\theta)(1-a(\theta))|x=1\right] R_F = I_0 + I_1,$$

and

$$\underline{D}E\left(\theta|x=0\right) = I_0.$$

[5] In fact, a pooling equilibrium in which all projects are refinanced cannot hold if for some projects the need of refinancing is sufficiently unlikely, that is, if $\overline{\theta}$ is close to 1. A pooling equilibrium in which no projects are refinanced does not exist if the need of refinancing is sufficiently high for some projects, that is, if $\underline{\theta}$ is close to zero.

When these contracts are offered, there exists a $\theta^* \in (\underline{\theta}, \overline{\theta})$ such that, if $\theta \geq \theta^*$, the second contract is chosen by the entrepreneur, while the first contract is picked otherwise. This equilibrium implies that only if the entrepreneur pays a sufficiently high amount to the bank he has access to a credit line. This follows because the entrepreneur takes actions that go against the bank's interests if those funds are provided. The premium charged *ex-ante*, in turn, will preclude the re-financing of some projects as the entrepreneur will only be willing to pay the premium when projects are more likely to need refinancing.

For the sake of comparison with the next sections, one should note that, whenever granted a credit line, the entrepreneur will use his private information – the value of θ – in a non trivial way when choosing the action to take. More specifically, in period 2, the entrepreneur solves

$$\max_{a \in [0,1]} (1-\theta) a R_E - c(a).$$

Denote by $a^{NE}(\theta)$ the unique solution of this problem. Clearly, $a^{NE}(\theta)$ is continuous. Moreover, since the objective function has increasing differences in $(a, -\theta)$ and the solution is interior, $a^{NE}(\theta)$ is strictly decreasing in θ (Topkis [1998]). Hence, the higher the likelihood of refinancing being needed, the more the entrepreneur will act against the bank's interest.

4 Relationship Lending

4.1 Preliminaries: Strategies and Payoffs

Throughout a relationship, both the entrepreneur and the bank can condition present behavior (e.g., whether refinancing is provided) on what has been observed in the past. Using this observation, we model a lending relationship as a Public Pure Equilibrium (PPE) of the repeated game played by both parties.

To define strategies and the players' payoff in the repeated game, we need to establish what is publicly observed by the parties. A public history of length 1 (the initial history), h^0, is just the empty set. For $t > 0$, a public history of length t, h^t, is a sequence of

1. contracts offered by the Bank to the entrepreneur, $\left\{\varnothing, \{D_1, x_1\}_{x_1 \in \{0,1\}}, ..., \{D_{t-1}, x_{t-1}\}_{x_{t-1} \in \{0,1\}}\right\}$
2. actions $\{\varnothing, a_1, ..., a_{t-1}\}$ taken by the entrepreneur, and, finally,
3. outcomes of a public random device observed before the players act at each stage of the game, $\{\varnothing, \varpi_1, ..., \varpi_{t-1}\}$, which serve the purpose of convexifying the set of equilibrium payoffs in the repeated game.

At each period t, the bank and the entrepreneur observe a history h^t, and can condition their current and future behavior on h^t. Letting H_t be the set of all possible public histories of length t, a strategy for the entrepreneur is a sequence of functions $\{a_t(.)\}_{t=1}^{\infty}$, where, for each t,

$$a_t : H_t \times [\underline{\theta}, \overline{\theta}] \to \Re.$$

A strategy for the Bank is a sequence of contracts to be offered where, for each t, the contract offered is a function of the history h^t.[6]

[6] Banks not engaged in a relationship with the borrower will offer the arm's length contract at every h^t and all t in equilibrium.

At period t, given the the contract $\{D_t, x_t\}$ picked and the action taken, a_t, the entrepreneur's instantaneous utility is given by

$$u_t(a_t, D_t, x_t; \theta) = \theta_t(R - D_t) + (1 - \theta_t)x_t a_t R_E - c(a_t).$$

Along an infinite history, the entrepreneur's payoff is the discounted sum of his instantaneous utility,

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t u_t(a_t, D_t, x_t; \theta).$$

The entrepreneur's and the bank's strategies define a probability distribution over public histories in the usual way, so the entrepreneur's payoff is

$$E\left[(1-\delta)\sum_{t=0}^{\infty}\delta^t u_t(a_t, D_t, x_t; \theta)\right],$$

where the expectation is taken with respect to the probability distribution induced by the strategy profiles.

4.2 Computing the Set of PPE's Payoff

4.2.1 A Recursive Formulation

We solve this game by applying the recursive methods developed by Abreu [1988] and Abreu, Pearce and Stachetti [1990]. More specifically, letting $V \subset \Re$ be the set of Public Pure Equilibria (PPE) Payoffs for the entrepreneur[7], any v in V can be written as

$$v = E[(1-\delta)u(a, D, x; \theta) + \delta v(a)],$$

where $v(a) \in V$ for all $a \in \Re$.

In words, any PPE can be summarized by a set of period 1 contracts $\{D, x\}_{x\in\{0,1\}}$, an action a to be taken in the initial period, and promised continuation values, $v(a)$, which depend on the action taken and incorporates all relevant aspects of the future play between the entrepreneur and the bank.

We search for the best lending relationship, that is, the action a, the contract $\{D, x\}_{x\in\{0,1\}}$, and continuation values that solve

$$\max_{\{D,x\}_{x\in\{0,1\}}, a, \{v(a')\}_{a'\in\Re}} E\left[(1-\delta)u(a, D, x; \theta) + \delta v(a)\right] \qquad \text{(Program)}$$

subject to

$$(1-\delta)u(a, D, x; \theta) + \delta v(a) \geq (1-\delta)u(a', D, x; \theta) + \delta v(a') \text{ for all } a', \qquad \text{(IC)}$$

$$v(a') \in V \text{ for all } a', \qquad \text{(Feasibility)}$$

and

This guarantees that borrowers will not abandon their relational bank, i.e., they will not deviate and try to start a relationship with another bank. Also, offering the arm's length contract is, given the strategies of the other players, optimal for the banks not engaged in the lending relationship.

[7] This set is not empty as the infinite repetition of the actions taken in the static game constitutes a PPE equilibrium.

$$D=\begin{cases}\overline{D}\text{ if }x=1\\ \underline{D}\text{ if }x=0\end{cases}, \tag{Zero-Profit}$$

where $\overline{D}$ satisfies

$$\overline{D}E(\theta|x=1)+E\left[(1-\theta)(1-a(\theta))|x=1\right]R_F=I_0+I_1,$$

while $\underline{D}$ satisfies

$$\underline{D}\ E(\theta|x=0)=I_0.$$

The first constraint states that the prescribed action is incentive compatible given continuation values $v(.)$. The second constraint calls for continuation values $v(a')$ being PPE values for any action a'. The last constraint is the zero-profit condition for the bank.

Toward solving (Program), and following Athey et al [2004], it is useful to decompose the incentive compatibility constraints into two different sets of constraints. An entrepreneur with project θ can think about deviating to two different set of actions. First, he can deviate and take an action $a\left(\widehat{\theta}\right)$ that would be prescribed for him if he had a project $\widehat{\theta}\neq\theta$. Second, he can deviate and take an action $a'\notin a\left(\left[\underline{\theta},\overline{\theta}\right]\right)$ which is not prescribed to be taken on the equilibrium path.

The Incentive Compatibility constraints in (IC) can be then equivalently written as

$$(1-\delta)u(a(\theta),D,x;\theta)+\delta v(a(\theta))\geq(1-\delta)u(a(\widehat{\theta}),D,x;\theta)+\delta v(a\left(\widehat{\theta}\right)),\text{ for all }\theta,\widehat{\theta} \tag{IC1}$$

and, for all $a'\notin a\left(\left[\underline{\theta},\overline{\theta}\right]\right)$,

$$(1-\delta)u(a(\theta),D,x;\theta)+\delta v(a(\theta))\geq(1-\delta)u(a',D,x;\theta)+\delta v(a'). \tag{IC2}$$

Given (IC1) and (IC2), the feasibility constraints can be re-written as: $v(a\left(\widehat{\theta}\right)),v(a')\in V$ for all $\widehat{\theta},a'\notin a\left(\left[\underline{\theta},\overline{\theta}\right]\right)$.

4.2.2 Properties of the Set of PPE Values

We now derive some important properties of the set of equilibrium payoffs, V.

For a given set $W\subset R$, consider the following operator

$$T(W)=\left\{\begin{array}{l} v|\text{ there exists }\{D,x\}_{x\in\{0,1\}},\ \{a(\theta)\}_\theta,\text{ and }\{w(a)\}_{a\in\Re}\subset W,\text{ satisfying the restrictions (IC1), (IC2),}\\ \text{the Zero-Profit condition in Program, and so that }v=E_\theta[(1-\delta)u(a(\theta),D,x;\theta)+\delta w(a(\theta))]\end{array}\right..$$

Following APS's jargon, a v in $T(W)$ is said to be enforceable by W. APS [1990] have shown that the set of PPE's Payoff is given by the largest fixed point of this operator; i.e., the V that satisfies $T(V)=V$ and so that, for all other Ω with $T(\Omega)=\Omega$, $\Omega\subset V$.

A slight complication arises because our setting differs from the one in APS in an important dimension: the actions taken by the entrepreneur lie in a continuous set. In order to argue that a maximum for (Program) exists, we need to show that $T(.)$ takes compact sets into compact sets.

We use the particular structure of the constraints implied by $T(.)$, the similarity of those constraints with the ones arising in standard static mechanism design problems, and the entrepreneur's payoff to do so.

We start with the following result, which is derived from an application of the Envelope Theorem (Milgrom and Segal, [2002]) to the Incentive Compatibility constraints implied by (IC1)

Lemma 1 (IC Representation) *Fix a set $W \subset \Re$. Any v in $T(W)$ can be written as*

$$\begin{aligned} v &= E_\theta[(1-\delta)u(a(\theta), D, x; \theta) + \delta w(a(\theta))] \\ &= (1-\delta)\, u(a(\overline{\theta}), D, x; \overline{\theta}) + \delta w(a(\overline{\theta})) + (1-\delta)\, E_\theta \left[(x\, [a(\theta)\, R_E] - [R - D]) \frac{F(\theta)}{f(\theta)} \right], \end{aligned}$$

where $xa(\theta)$ is non-increasing in θ.

Using the IC Representation Lemma, we can show the following result

Lemma 2 ($T(.)$ preserves compactness) *$T(.)$ takes compact sets into compact sets.*

As shown by [1990], an immediate implication of the fact that the operator $T(.)$ preserves compactness is that the set of PPE values V, which is the largest fixed point of $T(.)$, is itself compact. We then have

Proposition 1 *Among all PPE payoffs, there exists a smallest payoff, $\underline{v}$, and a largest payoff, $\overline{v}$. Hence, there exists a solution for (Program).*

We can now fully characterize the (implicit) contract that leads to the best lending relationship between the entrepreneur and the bank. The form of this contract will depend on the entrepreneur's degree of patience. We first analyze the case in which the entrepreneur is patient (δ is large, potentially approaching 1) and then we move to the case in which his discount rate is strictly smaller than one.

4.3 The Case of a Patient Entrepreneur

Along the repeated interaction, when considering deviations from the behavior specified in an equilibrium, the entrepreneur faces two possibilities. First, he could take the action $a\left(\widehat{\theta}\right)$ prescribed for a project of quality $\widehat{\theta}$ when, in fact, if the true quality of the project is θ, he must take $a(\theta)$. Such a deviation would never be detected by the bank. Consequently, the constraints in (IC1) have to be taken into account no matter what the entrepreneur's discount rate is.

Second, she could take an action which is not prescribed to be taken in equilibrium for *any* possible project quality; that is an action $a \notin a([\underline{\theta}, \overline{\theta}])$. Such a deviation *is* detectable by the bank. Moreover, it is always optimal to make use of the harshest possible punishment to induce a non-deviating behavior from the entrepreneur (Abreu [1988]).

This last observation suggests that, when δ is sufficiently large, the constraints in (IC2) will not be binding. To see that this is indeed the case, consider, for instance, the punishment induced by Nash Reversal[8]. Letting V^{NE} be the stage game Nash (expected) payoff, $v(a(\theta))$ be the continuation value prescribed by the best PPE when the entrepreneur has project θ and takes the corresponding action takes the action $a(\theta)$, and a' any action which is not in $a([\underline{\theta}, \overline{\theta}])$. One has that, for all such a' and for all θ,

$$\frac{\delta}{1-\delta}[v(a(\theta))) - V^{NE}] \geq u(a', D; \theta) - u(a(\theta), D; \theta), \forall \theta$$

when δ is large and if $v(a(\theta))) > V^{NE}$.

[8] That is, the reversion to playing the static Perfect Bayesian Equilibrium indefinitely following an observable deviation by the entrepreneur. Since he has the option of searching for other banks (arm's length financing), this is the worst punishment that can be inflicted on the entrepreneur.

We show below that the best lending relationship prescribes, along the equilibrium path, continuation values that are higher than static Nash values for all projects, that is

$$v(a(\theta)) > V^{NE} \text{ for all } \theta.$$

Hence, when the entrepreneur is sufficiently patient, we can safely ignore the constraints in (IC2).

We now argue that the optimal lending relationship will be such that refinancing is provided for all projects. The reason for that is simple. Whenever feasible, it will be *optimal* to refinance all projects. In fact, since at each period the bank has zero profits, the entrepreneur is the residual claimant of the total surplus. As it is efficient to provide refinancing to all projects, and the best lending relationship maximizes the entrepreneur's *ex-ante* (i.e., before the realization of the project's quality) payoff, it is optimal for the entrepreneur to have all projects refinanced.

If refinancing is prescribed for all projects in a lending relationship, an entrepreneur who faces a project with high probability of going through without the need of refinancing, could be tempted to deviate and seek for lower repayments under arm's length financing. Such deviation, however, will be punished with arm's length financing for all future periods. Therefore, upon deviating, the entrepreneur's expected future payoff will be V^{NE}. Hence, the deviation is unprofitable if the discount factor is sufficiently high.

Proposition 2 *There exists a $\overline{\delta} < 1$ such that, if $\delta \geq \overline{\delta}$, refinancing is provided to all projects at an optimal lending relationship.*

Refinancing to all projects is one of the main features of relational lending in our model: repeated interaction between the entrepreneur and a bank arises exactly to curb the inefficiencies – lack of refinancing for some projects – related to a single interaction.

We have argued above that to satisfy (IC2), it is optimal to use the harshest possible punishment, V^{NE}, when the entrepreneur deviates to some $a' \notin a\left(\left[\underline{\theta}, \overline{\theta}\right]\right)$. We now consider how the constraints in (IC1) and optimality considerations shape the choice of continuation values for action in $a\left(\left[\underline{\theta}, \overline{\theta}\right]\right)$. To make the analysis, it is convenient to think of an equivalent formulation of the problem in which the entrepreneur reports his project type, θ, to the bank that, given the report, suggests an action to be taken by the by the entrepreneur.

If the optimal lending relationship prescribes different actions for different project types, continuation values must vary to provide incentives. As an example, suppose that the lending relationship calls for an action schedule which is, at the same type, smaller than $a^{NE}(\theta)$ (the action taken in the one-shot game), and strictly downward sloping. In such case, an entrepreneur who announces a higher type (i.e., takes a lower action) must be rewarded through higher continuation values. Analogously, if the entrepreneur announces a lower type (i.e., takes a higher action), he must be promised lower continuation values.

This suggests that, along the equilibrium path, continuation values will play a key role if the lending relationship calls for "separation" of different projects, that is, the entrepreneur taking different actions whenever facing different project types. It turns out, however, that at an optimum lending relationship calls for different projects being "pooled" together. This happens for two reasons.

To understand the first reason, it is useful to consider the implications of the action schedule $a^{NE}(\theta)$ that prevails under arm's length financing. In an one shot interaction, the bank demands a higher payment to refinance the project because of its needs to recoup the losses associated with the low *ex-post* returns arising from projects with high probability of needing additional funds. Given a project type θ, the *ex-ante* probability the bank has to recoup money in case refinancing is granted is $(1-\theta)(1-a^{NE}(\theta))$. Since $a^{NE}(\theta)$ is strictly decreasing and projects with lower θs are the ones seeking refinancing, in expectation, this probability ends up being small. In a repeated setting, the entrepreneur would like to reduce the bank's expected losses by increasing the likelihood of the financier's preferred state.

There are two ways in which this can be accomplished. The first and more direct one is by taking an action $a(\theta)$ which is smaller than the static optimal one for every project type θ. The second and more interesting one is by taking an action which *does not* vary with θ. The interpretation is straightforward. Given the lack of contractibility, the provision of additional funds to the entrepreneur grants him control over relevant decisions. The bank charges up-front for such a delegation of control. The amount charged is higher the more the entrepreneur exercises control. By choosing actions that do not depend on the project type, the entrepreneur refrains from doing it.

The second reason why the entrepreneur would like to refrain from adopting an action that depends on the realized project, θ, is a bit less intuitive: from an *ex-ante* perspective, the entrepreneur puts a higher weight on states for which she does not need refinancing. Therefore, she takes actions which are closer to the ones that are optimal for projects with high θs.

The formal description of both effects are as follows. Using Lemma 1, the objective function of (1) reads, after a few manipulations, as

$$(1-\delta)\left[\left(1-\overline{\theta}\right)a\left(\overline{\theta}\right)R_E-c\left(a\left(\overline{\theta}\right)\right)\right]+\delta v\left(a\left(\overline{\theta}\right)\right)+(1-\delta)\left[R_E E\left[a(\theta)\frac{F(\theta)}{f(\theta)}\right]+E(\theta)\left[R-D\right]\right],$$

where D is the payment that solves

$$DE(\theta)+E\left[(1-\theta)(1-a(\theta))\right]R_F=I_0+I_1.$$

Assume that, for some interval $(\theta_*,\theta^*)\subset\left[\underline{\theta},\overline{\theta}\right]$, $\frac{dv(.)}{d\theta}$ is, for example, positive. Incentive compatibility then requires that $\frac{da(\theta)}{d\theta}<0$ for all θ over this interval.[9] Consider replacing $a(\theta)$ by its expected value, $E\left[a(\theta)\,|\,(\theta_*,\theta^*)\right]$, over such interval. We argue that an improvement could be attained in two fronts.

The first one is through a reduction in the payment charged by the bank. More precisely, since $\frac{da(\theta)}{d\theta}<0$ over (θ_*,θ^*), and $(1-\theta)$ is decreasing,

$$\begin{aligned}&E\left[(1-\theta)(1-a(\theta))\,|\theta_*,\theta^*\right]R_F\\ <\ &E\left[(1-\theta)\right]|\,(\theta_*,\theta^*)\left[1-E\left[a(\theta)|\,(\theta_*,\theta^*)\right]\right]R_F.^{10}\end{aligned}$$

This last inequality says that the replacement of $a(\theta)$ by $E\left(a(\theta)\,|\,(\theta_*,\theta^*)\right)$ in (θ_*,θ^*) increases the likelihood of the states in which the bank recoup some money in case refinancing is provided. This, in turn, implies that the required repayment D will be smaller, which will lead to gains. Note that a lower repayment made to the bank is beneficial to *all* project types. Therefore, the gains we just derived do not rely on any distributional assumption on the types of projects.

Now, consider the term

$$(1-\delta)\left[R_E E\left(a(\theta)\frac{F(\theta)}{f(\theta)}\right)\right]$$

[9] In the appendix, we consider the case in which $v(.)$ is not differentiable.

of the objective function.

Since $\frac{F(\theta)}{f(\theta)}$ is non-decreasing,

$$\begin{aligned} & E\left[a(\theta)\frac{F(\theta)}{f(\theta)}|\left(\theta_*,\theta^*\right)\right] \\ < \; & E\left[a(\theta)|\left(\theta_*,\theta^*\right)\right]E\left[\frac{F(\theta)}{f(\theta)}|\left(\theta_*,\theta^*\right)\right]. \end{aligned}$$

Hence, the replacement of $a(\theta)$ by $E(a(\theta)|(\theta_*,\theta^*))$ in (θ_*,θ^*) also generates gains for the term $R_E E\left[a(\theta)\frac{F(\theta)}{f(\theta)}\right]$ of the objective.

As the action prescribed for project $\overline{\theta}$ was not affected, if the proposed change is feasible, it increases the objective. Feasibility calls for existing continuation values in V that guarantee that the change is incentive compatible. In the Appendix, we show that one can indeed find continuation values in V that guarantee that the proposed change is Incentive Compatible.

This discussion suggests that, on the equilibrium path, $v(a(\theta))$ should be constant at an optimum. Among the constant continuation values, the best one is the highest.[11] We then haven

Proposition 3 *At an optimum of (Program),*

$$v(a)=\begin{cases} \overline{v} \text{ if } a\in a\left(\left[\underline{\theta},\overline{\theta}\right]\right) \\ v^{NE} \text{ if } a\notin a\left(\left[\underline{\theta},\overline{\theta}\right]\right)\end{cases}$$

.

The result shows that, on the equilibrium path, an optimal lending relationship is stationary. Along with the fact that refinancing is granted to all project types, Proposition 3 implies that the implicit contract between the lender and the entrepreneur satisfies a strong form of renegotiation-proofness: current (as all projects are refinanced) as well as future (as only the highest continuation value is promised to the entrepreneur) efficiency is attained. This is in sharp contrast to "screening" models in which some sort of relationship drives lending (e.g., Bolton and Scharfstein [1990]).

Using Proposition 3, $\overline{v}$ can be expressed as

$$\overline{v}=\max_{\{a(\theta)\}}\left[\left(1-\overline{\theta}\right)a\left(\overline{\theta}\right)R_E-c\left(a\left(\overline{\theta}\right)\right)+\left[R_E E\left(a(\theta)\frac{F(\theta)}{f(\theta)}\right)+E(\theta)\left[R-D\right]\right]\right], \tag{P2}$$

subject to $a(\theta)$ being incentive compatible, and D guaranteeing zero profits for the bank.

[11] Continuation values that do not depend on announcements are also optimal in Athey et al (2005), and Maggi and Morelli (2006). In Athey et al (2005), logconcavity is the (only) force leading to such a result. In our setting, an additional and particular force is present: an implicit contract that uses type dependent continuation values induces too a high of contractible repayment from the entrepreneur to the bank (i.e., too high of an interest rate). This effect holds true for *all* possible distributions so that the above result is likely to hold in our setting even for distributions which are not log-concave as long as the repayment demanded by the bank is large enough, which happens, for example, if R_F and R are relatively low. In Maggi and Morelli's discrete type model, the result is a consequence of the particular structure that defines how a collective action is taken. In such a structure, truthful reporting is never binding so continuation values are not needed to ensure truthtelling and, of course, it does not pay to use continuation values other the highest ones for all states.

4.3.1 The Optimal Lending Relationship

Full characterization of the optimal lending relationship involves finding the action schedule that solves (P2). When choosing the optimal schedule, one has to consider three effects. The first one is the effect of the schedule on the payment, D, to be made from the entrepreneur. The second is the effect of the chosen action on the "informational rents",

$$E_\theta \left[[(a(\theta) R_E] \frac{F(\theta)}{f(\theta)} \right],$$

component of the objective, while the third effect is on the utility of the entrepreneur when he has the best possible project, $\overline{\theta}$.

In the previous section, we showed that the first two effects alone would lead to actions that do not respond to the types of project that the entrepreneur faces. This, however, can potentially hurt type $\overline{\theta}$. Hence, the optimal action schedule trades-off the effect of the schedule on $\overline{\theta}$ and the need for constancy.

There are a number of interesting cases in which it will be optimal to have a constant action. In fact, the relevant condition on the parameters for that to happen is that

$$R_E \left[\overline{\theta} - E(\theta) \right] \leq R_F \left[1 - E(\theta) \right]. \tag{A2}$$

To see why this is the case, note that the objective can be written as

$$\underbrace{\left(1-\overline{\theta}\right) a\left(\overline{\theta}\right) R_E - c\left(a\left(\overline{\theta}\right)\right)}_{A} + \underbrace{E(\theta)\left[R - D\right] + R_E E\left[a(\theta) \frac{F(\theta)}{f(\theta)}\right]}_{B}. \tag{1}$$

Consider first term B. As argued in the previous section, since for all non-increasing schedules one has that

$$E\left[(1-\theta)(1-a(\theta))\right] R_F \leq E\left[(1-\theta)\right] E\left[(1-a(\theta))\right] R_F$$

(so that a constant action maximizes the likelihood of the banks most favorable states in case of refinancing which, in turn, results in lower payments from the entrepreneur to the bank) and

$$E\left[a(\theta) \frac{F(\theta)}{f(\theta)}\right] \leq E\left[a(\theta)\right] E\left[\frac{F(\theta)}{f(\theta)}\right],$$

term B is maximized by the choice of a constant action.

This constant action balances the negative impact of a higher action on the payment to be made to the bank when the project succeeds, D, with the benefit of higher informational rents. In other words, for term (B), the relevant trade-off is between the premium for refinancing charged by the bank and the informational rents for types below $\overline{\theta}$.

By the implicit function theorem, for any constant action, the cost of a higher a on the term that depends on D is

$$E(\theta) \frac{\partial D}{\partial a} = E(\theta) R_F \frac{E[(1-\theta)]}{E(\theta)} = R_F \left[1 - E(\theta)\right].$$

The benefit associated with the informational rents of a higher a is

$$R_E E\left[\frac{F(\theta)}{f(\theta)}\right] = R_E \left[\overline{\theta} - E(\theta)\right].$$

Under (A2), the marginal cost of a higher action is higher than the marginal benefit, so the constant action schedule that maximizes term B is $\underline{a}$.

Recall that term A in (1) is

$$\left(1-\overline{\theta}\right) a\left(\overline{\theta}\right) R_E - c\left(a\left(\overline{\theta}\right)\right),$$

which would be maximized by the choice of any action schedule $a(.)$ with $a\left(\overline{\theta}\right) = a^{NE}\left(\overline{\theta}\right)$, where

$$a^{NE}\left(\overline{\theta}\right) = \arg\max \left(1-\overline{\theta}\right) a\left(\overline{\theta}\right) R_E - c\left(a\left(\overline{\theta}\right)\right),$$

the static best action for the entrepreneur when his project is $\overline{\theta}$.

Now, consider an arbitrary non-constant schedule $\widetilde{a}(.)$ which is incentive compatible. If one replaces $\widetilde{a}(.)$ by

$$a(\theta) = \widetilde{a}\left(\overline{\theta}\right) \text{ for all } \theta,$$

an improvement can be attained. In fact, the term

$$\left(1-\overline{\theta}\right) a\left(\overline{\theta}\right) R_E - c\left(a\left(\overline{\theta}\right)\right)$$

is not affected, whereas a strict gain is brought up for the term

$$E(\theta)[R-D] + R_E E\left[a(\theta)\frac{F(\theta)}{f(\theta)}\right]$$

since $a(\theta)$ is closer to $\underline{a}$ (the action that maximizes this term) for a positive probability set of projects.

This discussion proves

Proposition 4 *The best lending relationship when (A2) holds and δ is large is such that $a(\theta) = a$ for all θ, where a satisfies*

$$[1-E(\theta)][R_E - R_F] - c'(a) \leq 0,$$

with equality if $a > \underline{a}$.

Under (A2), in spite of having control over the decisions to take, the entrepreneur completely refrains from exercising it by committing to a single action irrespective of the quality of his project. Alternatively, the bank completely eliminates the entrepreneur's discretion over decisions, demanding him to take a given action throughout the relationship.

A couple of interesting observations can be made using the inequality that characterizes the optimal action under (A2). First, $a < a^{NE}\left(\overline{\theta}\right) \leq a^{NE}(\theta)$ for all θ. In our model, a relationship generates an *endogenous* commitment from the entrepreneur's part to not act against the bank's interest. In equilibrium, not only the entrepreneur chooses an action schedule that does not respond to his private information but also commits to an action which is smaller than his best static action for *all* possible types. As a consequence of such commitment, the payment promised to the bank is

$$D(a) = \frac{I_0 + I_1 - (1-a)(1-E(\theta))R_F}{E(\theta)},$$

which is smaller than the payment in the arm's length financing corresponding to the static equilibrium.

Second, if $R_E \leq R_F$, the entrepreneur will choose the bank's most preferred action, $\underline{a}$. The interpretation for this is straightforward. If $R_E \leq R_F$, it is inefficient to take actions that are higher than $\underline{a}$. The repetition of the game – which serves as a commitment device to the entrepreneur – along with the fact that a higher action would induce higher interest rates guarantee that the entrepreneur fully internalizes the inefficiency costs associated with higher actions.

A low premium for refinancing is key in our relationship lending model. The disciplinary role of a relationship allows for a more lenient credit line policy from the bank at an interest rate which is *lower* than in the arm's length market, which rationalizes the received evidence in empirical banking literature (Berger and Udell [1995], for example). This is in contrast to learning explanations of relational lending in which firms strong banking relationships do not necessarily have access to credit at lower interest rates.

We now briefly discuss of some features of the optimal lending relationship when (A2) does not hold. In this case, the solution is a bit more complicated. To understand why it is the case, it is worth considering again the two terms in (1). The action schedule that maximizes the term B of the objective is $a(\theta) = 1 > a^{NE}(\theta) > a^{NE}(\overline{\theta})$. Hence, the choice of an action schedule which is closer to 1 may significantly harm the utility of the entrepreneur when his type is $\overline{\theta}$. The balance between the type $\overline{\theta}$'s payoff and the informational rents component of the objective may therefore call for a non-constant action schedule.

While it is hard to make a full blown characterization of the optimal action schedule, it turns out that, even when the forces that drive the optimal lending relationship toward the adoption of a single action conflict with the type $\overline{\theta}$'s payoff, it is optimal for the entrepreneur to refrain, at least to some degree, from using its private information against the bank. More precisely, at optimal lending relationship, the action schedule will be a step function.

Proposition 5 *When (A2) does not hold, there exists a finite number of actions $\{a_i\}_{i=1}^N$ and a partition of $[\underline{\theta}, \overline{\theta}]$ into finite intervals $\{A_i\}_{i=1}^N$ such that, at an optimal lending relationship, $a(\theta) = a_i$ if $\theta \in A_i$.*

Propositions 4 and 5 say that, at an optimal lending relationship under contractual incompleteness, the bank imposes to the entrepreneur a finite set of decisions from which he can pick his action, restricting substantially his discretion. The reduction in the entrepreneur's discretion to take actions minimizes the ex-post conflicts brought up by the contractual incompleteness.

4.4 The Case of an Impatient Entrepreneur

An impatient entrepreneur may be tempted, if asked to take an action that is too low given the realization of the project, to renege on the implicit contract prescribed by the lending relationship and take a higher action. In this section, we consider this possibility. We focus attention on the case in the temptation to deviate is the greatest, namely, we consider a situation in which (A2) holds.[12]

To understand better the new forces at play, consider the entrepreneur's incentives when facing a project of quality $\underline{\theta}$, and being called to adopt action the action $a < a^{NE}(\overline{\theta})$ from Proposition 4. By deviating and adopting the static best action, the entrepreneur's immediate gain is[13]

$$[(1-\underline{\theta})R_E a^{NE}(\underline{\theta}) - c(a^{NE}(\underline{\theta}))] - [(1-\theta)R_E a - c(a)].$$

[12] We also assume that, although not infinitely patient, the entrepreneur's discount rate is so that it is feasible to have all projects refinanced.

[13] We focus on $\underline{\theta}$ because, using the Envelope Theorem, the immediate gain from deviating as a function of θ is

$$R_E\left(a - a^{NE}(\theta)\right) < 0.$$

This deviation is punished in the harshest possible way by the bank. Given the fact that the entrepreneur has the option of searching for other banks (arm's length financing), the worst punishment is the indefinite repetition of the static equilibrium for all periods after the deviation, i.e., V^{NE}.

Therefore, observable deviations will not be deterred if, and only if

$$\frac{\delta}{1-\delta}[\overline{v} - V^{NE}] < [(1-\underline{\theta})R_E a^{NE}(\underline{\theta}) - c(a^{NE}(\underline{\theta}))] - [(1-\underline{\theta})R_E a - c(a)] \tag{1}$$

where $\overline{v}$ is the expected discounted sum of the entrepreneur's payoff, and a is the optimal constant action under the relationship implied by Proposition 4.

Under (1), the lending relationship needs to adjust the required action upward to decrease the gains from a deviation. This adjustment, however, has a perverse effect. By moving the action adopted away from a, the value of the lending relationship itself will fall, and this will reduce the perceived cost of a deviation.

The upward adjustment of the chosen action must then be made so to guarantee an exact equality between the costs and benefits that the entrepreneur faces when deviating. We now proceed to construct such an adjustment.

For a given constant $\widetilde{a}$, define

$$V(\widetilde{a}) \equiv \int_{\underline{\theta}}^{\overline{\theta}} [\theta[R - \widetilde{D}] + (1-\theta)\left(R_E\widetilde{a} - c(\widetilde{a})\right)] f(\theta) d\theta,$$

as the expected discounted value of a relationship when the constant action $\widetilde{a}$ is adopted, where $\widetilde{D}$ solves

$$E(\theta)\widetilde{D} + (1-\widetilde{a})E(1-\theta) - I_0 - I_1 = 0.$$

Assuming there exists $\widetilde{a}$ so that[14]

$$\frac{\delta}{1-\delta}[V(\widetilde{a})) - V^{NE}] \geq [(1-\underline{\theta})R_E(a^{NE}(\underline{\theta}) - \widetilde{a}) - (c(a^{NE}(\underline{\theta})) - c(\widetilde{a})],$$

let $a^*(\delta) \leq \widetilde{a}$ be the lowest action for which the temptation to deviate equals the future benefit to abide to the relationship,

$$\frac{\delta}{1-\delta}[V(a^*(\delta))) - V^{NE}] = [(1-\underline{\theta})R_E(a^{NE}(\underline{\theta}) - a^*(\delta)) - (c(a^{NE}(\underline{\theta})) - c(a^*(\delta))].$$

We will then have

Proposition 6 *If $a^*(\delta) \leq a^{NE}\left(\overline{\theta}\right)$, the best lending relationship when* 1 *holds has $a(\theta) = a^*(\delta)$ for all θ.*

The interpretation is straightforward. The choice of $a^*(\delta)$ is the one that gets the relationship, when constrained to have an impatient entrepreneur, closer to what would be the "unconstrained" optimal lending relationship.

It remains to answer what to do if $a^*(\delta) > a^{NE}(\overline{\theta})$. For such case, for some subset $[\theta', \overline{\theta}]$, one would have $a^*(\delta) > a^{NE}(\theta) > a$ for $\theta \in [\theta', \overline{\theta}]$. Therefore, the adoption of $a^*(\delta)$ for those projects would involve getting further away from the unconstrained optimum. An alternative would be to allow that types in $[\theta', \overline{\theta}]$ to

[14] If there is no such a', then there exists a $\theta' \leq \overline{\theta}$ such that all action schedules that can possibly prevail must equal the Static Nash one for $[\underline{\theta}, \theta']$.

take their static best action. However, although by making the action taken by these types closer to a, a benefit is attained, the fact that $a^{NE}(.)$ is strictly decreasing would impose – through a reduction in the informational rents provided to types below $\overline{\theta}$, and through a higher premium for refinancing – a cost to the relationship. The best way to reconcile the need to move closer to the unconstrained optimum $a(\theta) = a$ with an action schedule which does not allow for the use of the entrepreneur's private information against the bank is through a step function.

Proposition 7 *The best lending relationship when $a^*(\delta) \in \left(a^{NE}(\overline{\theta}), a^{NE}(\underline{\theta})\right)$ calls for the adoption of a schedule $a(\theta)$ which is a step function. That is, there exists a finite number of actions $\{a_i\}_{i=1}^N$ and a partition of $\left[\underline{\theta}, \overline{\theta}\right]$ into finite intervals $\{A_i\}_{i=1}^N$ such that, at an optimal lending relationship,*

$$a(\theta) = a_i \text{ if } \theta \in A_i.$$

The presence of an impatient entrepreneur hinders the lending relationship if it calls for an action which is too close to the Bank's most preferred one. In such a case, when facing a project which is likely to need refinancing, the entrepreneur will be tempted to renege on the implicit contract he holds with the bank.

In response to that, the required action will be adjusted toward one which is closer to the entrepreneur's static best action. Propositions 6 and 7 show that it will remain true, however, that the entrepreneur will be called, to the extent of what is feasible, to commit toward not using his private information against the bank.

5 Conclusion

We considered a model in which repeated lending plays a disciplinary role in a setting with incomplete contracts. Our main results are as follows. Under a one-shot interaction (arm's length financing), given the lack of contractibility, the bank charges up-front a premium whenever a contract that prescribes refinancing. As a consequence, projects that would be refinanced in a complete contract world will be shut down.

Repeated lending aligns the interests between lenders and borrows, allowing for refinancing of all projects. On the equilibrium path, the optimal lending relationship is stationary: following any action prescribed to be taken by the entrepreneur in equilibrium, he is promised the highest possible continuation values. Off-equilibrium, the entrepreneur gets the lowest possible continuation value; the one associated with the indefinite repetition of arm's length contracting. Along with refinancing being granted to all projects, the fact that the entrepreneur is promised the highest possible continuation values at each period implies that the implicit contract between the lender and the entrepreneur satisfies a strong form of renegotiation-proofness: current (as all projects are refinanced) as well as future (as only the highest continuation value is promised to the entrepreneur) efficiency is attained.

A robust feature of an optimal lending relationship is that the action schedule (as a function of project types) adopted by the entrepreneur is either a constant or a step function. Hence, the bank imposes to the entrepreneur a finite set of decisions from which he can pick his action, bounding his discretion over decisions.

References

[1] Abreu, Dilip, "On the Theory of Infinitely Repeated Games with Discounting", *Econometrica*, Vol. 56 (1988), No. 2, pp. 383-396.

[2] Abreu, Dilip, David Pearce and Ennio Stacchetti, "Toward a Theory of Discounted Repeated Games with Imperfect Monitoring", *Econometrica*, Vol. 58 (1990), No. 6, pp. 1041-1063.

[3] Aoki, Masahiko and Serdar Dinç, "Relational Financing as an Institution and its Viability under Competition," in *Finance, Governance, and Competitiveness in Japan*, Masahiko Aoki and Gary Saxonhouse, eds., Oxford: Oxford University Press, 2000.

[4] Aoki, Masahiko, Hugh Patrick and Paul Sheard, "The Japanese Main Bank System: an Introductory Overview," in *The Japanese Main Bank System*, Masahiko Aoki and Hugh Patrick, eds., Oxford: Oxford University Press, 1995.

[5] Athey, Susan, Kyle Bagwell, and Cris Sanchirico, "Collusion and Price Rigidity", *Review of Economic Studies*, 71 (2) (2004), pp.317-349.

[6] Athey, Susan, Andrew Atkenson, and Patrick Kehoe, "The Optimal Degree of Monetary Policy Discretion". Econometrica 73 (5), September (2005), pp. 1431-1476.

[7] Baums, Theodor, "The German Banking System and its Impact on Corporate Finance and Governance," in *The Japanese Main Bank System*, Masahiko Aoki and Hugh Patrick, eds., Oxford: Oxford University Press, 1995.

[8] Berger, Allen and Gregory Udell, "Relationship Lending and Lines of Credit in Small Firm Finance," *The Journal of Business*, Vol. 68 (1995), No 3, pp. 351-381.

[9] Bolton, Patrick and David Scharfstein, "A Theory of Predation Based on Agency Problems in Financial Contracting," *The American Economic Review*, Vol. 80, No. 1 (1990), pp. 1-25.

[10] Boot, Arnoud and Anjan Thakor, "Can Relationship Banking Survive Bank Competition?," *The Journal of Finance*, Vol. 55 (2000), No 2, pp. 679-713.

[11] Boot, Arnoud and Anjan Thakor, "Moral Hazard and Secured Lending in an Infinitely Repeated Credit Market Game," *International Economic Review*, Vol. 35 (1994), No. 4, pp. 899-920.

[12] Hoshi, Takeo, Anil Kashyap, and David Scharfstein, "Corporate Structure, Liquidity, and Investment: Evidence from Japanese Industrial Groups," *Quarterly Journal of Economics*, Vol 106 (1991), pp. 33-60.

[13] Kolmogorov, A., and Fomin, S., *Introductory Real Analysis.* New York, Dover Publications, 1970.

[14] Maggi, G. and Massimo Morelli, "Self-Enforcing Voting in International Organizations", *American Economic Review,* Vol. 96 (2006), No4, pp. 1137-1158.

[15] Mester, Loretta , Leonard Nakamura and Micheline Renault, "Transactions Accounts and Loans Monitoring," Working Paper No. 05-14 (2005), Research Department, Federal Reserve Bank of Philadelphia.

[16] Milgrom, Paul and Ilya Segal, “Envelope Theorems for Arbitrary Choice Sets,” *Econometrica*, Vol. 70 (2002), No. 2, pp. 583-601.

[17] Petersen, Michael and Raghuram Rajan, “The Effect of Credit Market Competition on Lending Relationships,” *The Quarterly Journal of Economics*, Vol. 110 (1995) , pp. 407-443.

[18] Petersen, Michael and Raghuram Rajan, “The Benefits of Lending Relationships: Evidence from Small Business Data,” *The Journal of Finance* Vol. 49 (1994), No 1, pp. 3-37.

[19] Rajan, Raghuram, “Insiders and Outsiders: The Choice between Informed and Arm’s Length Debt,” *The Journal of Finance*, Vol. 47 (1992), No. 4, pp. 1367-1400.

[20] Sharpe, Steven, “Asymmetric Information, Bank Lending and Implicit Contracts,” *The Journal of Finance* Vol. 45 (1990), No. 4, pp. 1069-1087.

[21] Rothschild, Michael and Joseph Stiglitz, “Equilibrium in Competitive Insurance: an Essay on the Economics of Imperfect Information,” *The Quarterly Journal of Economics*, Vol. 90 (1976), No. 4, pp. 629-649.

[22] Stiglitz, Joseph and Andrew Weiss, “Incentive Effects of Terminations: Applications to Credit and Labor Markets,” *The American Economic Review*, Vol. 73 (1983), No. 5, pp. 912-927.

[23] Topkis, D., *Supermodularity and Complementarity*, Princeton, New Jersey, Princeton University Press, 1998.

6 Appendix A: Proofs

Proof of Lemma 1: For any v in $T(W)$, there must exists contracts, an action schedule, and $\{w(\theta)\}_\theta \subset W$ so that

$$U(\theta) = (1-\delta)u(a(\theta), D, x; \theta) + \delta w(a(\theta)) \geq (1-\delta)u(a(\widehat{\theta}), D, x; \theta) + \delta w(a(\widehat{\theta})).$$

Standard Envelope Theorem arguments (Milgrom and Segal [2002]) along with the fact that the entrepreneur’s payoff has increasing differences in $(a, -\theta)$ imply that this will be true if, and only, if

$$U(\theta) = U(\overline{\theta}) + (1-\delta) \int_{\theta}^{\overline{\theta}} \left(x[(a(\widetilde{\theta})R_E] - [R-D] \right) d\widetilde{\theta}, \text{ and} \tag{Envelope}$$

$x(a(\theta))$ is decreasing in θ.

Taking the expectation of $U(\theta)$ – using integration by parts –, one has that

$$\begin{aligned} v &\equiv E_\theta \left[(1-\delta)u(a(\theta), D, x; \theta) + \delta w(a(\theta)) \right] \\ &= \begin{pmatrix} (1-\delta)u(a(\overline{\theta}), D, x; \overline{\theta}) + \delta w(a(\overline{\theta})) + \\ (1-\delta)E_\theta \left[x\left[a(\theta) R_E\right] - [R-D] \frac{F(\theta)}{f(\theta)} \right] \end{pmatrix} \end{aligned}$$

as claimed.■

Proof of Lemma 2: Fix a compact W, and let $\{v_n\}$ be a sequence in $T(W)$. We need to show that there exists a subsequence of $\{v_n\}$ which has a limit in $T(W)$.

From the definition of $T(W)$, there exists $\{w_n(.), D_n, x_n, a_n(.)\}$ so that

1. $v_n = (1-\delta)u(a_n(\overline{\theta}), D_n, x_n; \overline{\theta}) + \delta w_n(a_n(\overline{\theta})) + (1-\delta)E[x_n[a_n(\theta)R_E] - [R - D_n])\frac{F(\theta)}{f(\theta)}]$,

2. $x_n a_n(\theta)$ decreasing in θ for all n and x_n and D_n so that the Bank's profits are zero, and

3. for all $a' \notin a\left(\left[\underline{\theta}, \overline{\theta}\right]\right)$,

$$\left(\begin{array}{c} (1-\delta)u(a_n(\overline{\theta}), D_n, x_n; \overline{\theta}) + \delta w_n(a_n(\overline{\theta})) + \\ (1-\delta)\int\limits_{\theta}^{\overline{\theta}} \left(x_n[(a_n(\widetilde{\theta})R_E] - [R - D_n]\right) d\widetilde{\theta} \end{array}\right) \geq (1-\delta)u(a', D_n, x_n; \theta) + \delta w_n(a').$$

Note that, since W is compact, and $w_n(a_n(\overline{\theta}))$ is a sequence of real numbers in W, there exists a subsequence of $w_n(a_n(\overline{\theta}))$ which converges. By exactly the same reason, for a fixed a', there exists a subsequence of $w_n(a')$ which converges. Moreover, as $x_n a_n(\theta)$ is an uniformly bounded, decreasing function, by Helly's Selection Theorem (Kolmogorov and Fomin, 1970, p. 373), there exists a subsequence of which converges to a non-increasing function $xa(.)$. For each n, D_n, and x_n are sequences of real numbers in a compact set; there exists then convergent subsequences. Let the correspondent convergent subsequences be $\{w_{nk}(a_{nk}(\overline{\theta})), w_n(a'), a_{nk}(.), D_{nk}, x_{nk}\}$, and the limit be $\{w(a(\overline{\theta})), w(a'), a(.), D, x\}$.

We then have

$$v_{nk} = (1-\delta)u(a_{nk}(\overline{\theta}), D_{nk}, x_{nk}; \overline{\theta}) + \delta w_{nk}(a_{nk}(\overline{\theta})) + (1-\delta)E\left[x_{nk}\left[a_{nk}(\theta)R_E\right] - [R - D_{nk}]\frac{F(\theta)}{f(\theta)}\right].$$

Since it is continuous in the relevant variables, the term $(1-\delta)u(a_{nk}(\overline{\theta}), D_{nk}, x_{nk}; \overline{\theta}) + \delta w_{nk}(a_{nk}(\overline{\theta}))$ converges to $(1-\delta)u(a(\overline{\theta}), D, x; \overline{\theta}) + \delta w(a(\overline{\theta}))$. Moreover, since $\{a_{nk}(.)\}_{nk}$ is uniformly bounded, one can pick a large, but finite, positive number c such that

$$[x[a_{nk}(\theta)R_E] - [R - D_{nk}]]\frac{F(\theta)}{f(\theta)}$$

is also uniformly bounded by $cR_E\frac{1}{f(\overline{\theta})}$. Since $cR_E\frac{1}{f(\overline{\theta})}$ is integrable, one has, by Lebesgue's Dominated Convergence Theorem,

$$E[x_{nk}[a_{nk}(\theta)R_E] - [R - D_{nk}])\frac{F(\theta)}{f(\theta)}] \to E[x[a(\theta)R_E] - [R - D])\frac{F(\theta)}{f(\theta)}].$$

The above shows that $\{v_{nk}\}_k$ converges. Define its limit by v, it clearly satisfies the condition of Lemma 1.

It only remains to show that, in the limit, (IC2) is satisfied. Note that, for all n,

$$\left(\begin{array}{c} (1-\delta)u(a_n(\overline{\theta}), D_n, x_n; \overline{\theta}) + \delta w_n(a_n(\overline{\theta})) + \\ (1-\delta)\int\limits_{\theta}^{\overline{\theta}} \left(x_n[(a_n(\widetilde{\theta})R_E] - [R - D_n]\right) d\widetilde{\theta} \end{array}\right) \geq (1-\delta)u(a', D_n, x_n; \theta) + \delta w_n(a').$$

Moreover, along subsequences, the terms on the left hand side and on the right hand side converge (for the term on the left hand side, we again make use of the Dominated Convergence Theorem). Taking limits along those subsequences on both sides, it follows that

$$\left(\begin{array}{c} (1-\delta)u(a(\overline{\theta}), D, x; \overline{\theta}) + \delta w(a(\overline{\theta}))+ \\ (1-\delta) \int\limits_{\theta}^{\overline{\theta}} \left(x[(a(\widetilde{\theta})R_E] - [R-D] \right) d\widetilde{\theta} \end{array} \right) \geq (1-\delta)u(a', D, x; \theta) + \delta w\left(a'\right).$$

Hence, v is in $T(W)$ ■.

Proof of Proposition 1: Take a compact set $V_0 \subset \Re$ such that

$$V \subset V_0,$$

and

$$T(V_0) \subset V_0.$$

Define the decreasing (in the order induced by $\subseteq$) sequence $\{V_n\}_{n \geq 1}$ recursively as follows

$$V_n = T(V_{n-1}) \subset V_{n-1}.$$

The limit of this sequence is the largest fixed point of $T(.)$, which is V. Since $T(.)$ preserves compactness, V itself must be compact. Compactness of V implies that, among all PPE payoffs (which are elements of V), there exists a smallest payoff, $\underline{v}$, and a largest payoff, $\overline{v}$, as claimed.■

Proof of Proposition 2: It follows from the discussion in the text.■

Proof of Proposition 3: By Milgrom and Segal's Envelope Theorem, $(1-\delta)u(a(\theta), D; \theta) + \delta v(a(\theta))$ is almost everywhere differentiable. Therefore, at each point of differentiability, one must have,

$$\frac{d(1-\delta)u(a(\widehat{\theta}), D; \theta)}{d\widehat{\theta}}|_{\widehat{\theta}=\theta} + \frac{d\delta v(a(\theta))}{d\widehat{\theta}}|_{\widehat{\theta}=\theta} = 0.$$

As

$$\frac{d(1-\delta)u(a(\widehat{\theta}), D; \theta)}{d\widehat{\theta}}|_{\widehat{\theta}=\theta} = (1-\delta)[(1-\theta)R_E x - c'(a(\theta))]a'(\theta),$$

the sign of $\frac{d\delta v(a(\theta))}{d\widehat{\theta}}|_{\widehat{\theta}=\theta}$ will depend on how $a(\theta)$ compares to the action taken in the Static Nash Equilibrium.

Consider the case in which, for an open set of $\theta' s$, say $(\theta_*, \theta^*) \subset [\underline{\theta}, \overline{\theta}]$, for which the derivative holds $\delta \frac{dv(a(\theta))}{d\widehat{\theta}}|_{\widehat{\theta}=\theta} > 0$.[15] The text shows that, by replacing the prescribed actions $a(\theta)$ by $E(a(\theta)|\theta \in (\theta_*, \theta^*))$ for all θ over that set, and keeping the prescribed actions the same for all other θ, and improvement can be made.

We now show that such a change can be made Incentive Compatible. That is, letting

$$\widetilde{a}(\theta) = \left\{ \begin{array}{c} E(a(\theta)|\theta \in (\theta_*, \theta^*)) \text{ if } \theta \in (\theta_*, \theta^*) \\ a(\theta) \text{ otherwise} \end{array} \right. ,$$

we show that there exists feasible continuation values $\widetilde{v}(.)$ that make $\widetilde{a}(\theta)$ incentive compatible. (For what follows, and to save on notation, we express both $\widetilde{v}(.)$ and $v(.)$ – the original continuation values (that is, the ones that guarantee that $a(.)$ is incentive compatible) – as solely functions of the entrepreneurs' announcements).

[15] This implies that $a(\theta) < a^{NE}(\theta)$. The other case can be considered in an analogous fashion.

Assume first that

$$\Delta \equiv \int_{\theta_*}^{\theta^*} [E(a(\theta)|\theta \in (\theta_*, \theta^*)) - a(\tau)]d\tau \leq 0.$$

Define the continuation values $\widetilde{v}(.)$ as follows:

- for all $\theta \in [\theta^*, \overline{\theta}]$, let $\widetilde{v}(\theta) = v(\theta)$.
- for $\theta \in (\theta_*, \theta^*)$, let

$$\begin{aligned} \delta\widetilde{v}(\theta) &= \delta v(\theta) + (1-\delta)R_E \int_{\theta}^{\theta^*} [E(a(\theta)|\theta \in (\theta_*, \theta^*)) - a(\tau)]d\tau - \\ (1-\delta)[(1-\theta)R_E[E(a(\theta)|\theta &\in (\theta_*, \theta^*)) - a(\theta)] - c(a(\theta)) + c(E(a(\theta)|\theta \in (\theta_*, \theta^*)). \end{aligned}$$

- finally, for $\theta \leq \theta_*$ we let

$$\delta\widetilde{v}(\theta) = \delta v(\theta) + (1-\delta)R_E\Delta.$$

One can readily see that such continuation values guarantee that, given the new action profile, which is non-increasing, truthtelling is optimal for the entrepreneur. In fact, this is so because those continuation values were constructed so that the integral formula implied by the Envelope Theorem (the equation (Envelope) in the proof of Lemma 1) holds.

Moreover, for all θ, $\widetilde{v}(\theta) \leq \overline{v}$, so that the new continuation values are feasible. Indeed, (i) for all $\theta \in [\theta^*, \overline{\theta}]$, these values are feasible as they are exactly equal to the previous ones which are feasible by assumption, (ii) for all $\theta \leq \theta_*$, they are also feasible as they are equal to the previous continuation values added to a negative number. Finally, since for all θ in (θ_*, θ^*), $v'(\theta) < 0$, for all such θ it must be the case that $v(\theta)$ is bounded away from $\overline{v}$. By picking θ_*, and θ^* close enough from each other, using the continuity of the integral with respect to its bounds and the continuity of

$$R_E a - c(a)$$

in a, $\widetilde{v}(\theta)$ is feasible for such $\theta's$ as well.

If $\Delta \geq 0$, we can proceed in a slightly different way. If $\theta \in [\underline{\theta}, \theta_*]$, the continuation values are kept the same. If $\theta \in \left[\theta^*, \overline{\theta}\right]$ the new continuation values are equal to the previous ones minus Δ. For the remaining cases, the continuation values are similar to the ones constructed above. It can be easily shown that, such continuation values are feasible, and that the integral formula implied by the Envelope Theorem will hold with these continuation values, but now the "reference" type will be $\underline{\theta}$. More precisely, the following will hold

$$\begin{aligned} & (1-\delta)\left[(1-\theta)\,\widetilde{a}(\theta)R_E - c\left(\widetilde{a}(\theta)\right)\right] + \delta\widetilde{v}(\theta) \\ = & \left((1-\delta)\left[(1-\underline{\theta})\,\widetilde{a}(\underline{\theta})R_E - c\left(\widetilde{a}(\underline{\theta})\right)\right] + \delta\widetilde{v}(\underline{\theta}) - (1-\delta)\int_{\underline{\theta}}^{\theta} \widetilde{a}(\tau)R_E d\tau \right). \end{aligned}$$

This along with the fact that $\widetilde{a}(\theta)$ is non-increasing will imply Incentive Compatibility.

The discussion just shows that non-constant differentiable continuation values cannot be optimal.

Consider now the possibility of $v\,(.)$ jumping at some θ'. If that is the case, there is a positive measure set $[\theta_*, \theta^*] \subset [\underline{\theta}, \overline{\theta}]$ and a θ' in the interior of $[\theta_*, \theta^*]$ for which $\theta \leq \theta'$ implies that $v(\theta) = v_*$ and $\theta > \theta'$

implies that $v(\theta) = v^*$, with $v_* < v^* \leq \overline{v}$. This can only be Incentive Compatible if $a(\theta) = a_*$ for $\theta \leq \theta'$ and $a(\theta) = a^*$ for $\theta > \theta'$, with $a_* > a^*$ (if the actions taken were the same, and the continuation values were different, an entrepreneur with $\theta \leq \theta'$ would be strictly better off announcing $\widehat{\theta} > \theta'$). As in the text, if one substitutes $a(\theta)$ by $E(a(\theta)|\theta \in [\theta_*, \theta^*])$ for all θ over that set and make a strict improvement in the objective. If there is $\epsilon > 0$ so that $v_* < v^* \leq \overline{v} - \epsilon$, the same continuation values constructed for the differentiable case would do the job of making the change incentive compatible.

Otherwise, that is, when $v^* = \overline{v}$, replace $a(\theta)$ by

$$\widetilde{a}(\theta;\alpha) = \begin{cases} (1-\alpha)\,a(\theta) + \alpha E(a(\theta)|\theta \in [\theta_*, \theta^*]) \text{ if } \theta \in [\theta_*, \theta^*] \\ a(\theta) \text{ otherwise} \end{cases}.$$

By the standard reasons, for any $1 > \alpha > 0$, such replacement, if incentive compatible, will improve the objective. We start pointing out that this policy is feasible for $\alpha = 0$ as it is then exactly the $a(\theta)$ which is feasible by assumption.

Assume that

$$\widetilde{\Delta}(\alpha) \equiv \int_{\theta_*}^{\theta^*} [\widetilde{a}(\theta;\alpha) - a(\tau)]d\tau \leq 0. \text{ for all } \alpha,$$

and consider: $\widetilde{v}(\theta;\alpha) = v(\theta)$ for all $\theta \in [\theta^*, \overline{\theta}]$,

$$\begin{aligned} \delta\widetilde{v}(\theta;\alpha) &= \delta v(\theta) + (1-\delta)R_E \int_{\theta}^{\theta^*} [\widetilde{a}(\theta;\alpha) - a(\tau)]d\tau \\ &\quad -(1-\delta)[R_E(1-\theta)[\widetilde{a}(\theta;\alpha) - a(\theta)] - c(a(\theta)) + c(\widetilde{a}(\theta;\alpha)). \end{aligned}$$

for $\theta \in [\theta_*, \theta^*]$ and, finally,

$$\delta\widetilde{v}(\theta;\alpha) = \delta v(\theta) + (1-\delta)R_E\widetilde{\Delta}(\alpha) \text{ for } \theta < \theta_*.$$

Clearly, whenever $\theta \in [\underline{\theta}, \theta_*) \cup [\theta^*, \overline{\theta}]$, continuation values are feasible.

Moreover, for $\theta \in \left[\theta_*, \theta'\right]$, as $v(\theta) = v_* < v^* = \overline{v}$, by continuity, $\delta\widetilde{v}(\theta;\alpha)$ is feasible for α close enough to zero.

We will now show that, by choosing θ_* and θ^* properly, $\frac{d\delta v(\theta;\alpha)}{d\alpha}|_{\alpha=0} < 0$ for the remaining cases. Hence, as the continuation values are feasible when $\alpha = 0$, they will also be when α is positive but close enough to zero.

Note that

$$\begin{aligned} \frac{d\delta v(\theta;\alpha)}{d\alpha}|_{\alpha=0} &= (1-\delta)R_E \int_{\theta}^{\theta^*} [E(a(\theta)|\theta \in (\theta_*, \theta^*)) - a(\tau)]d\tau - \\ (1-\delta)[R_E(1-\theta) - c'(a(\theta))][E(a(\theta)|\theta &\in (\theta_*, \theta^*)) - a(\theta)]. \end{aligned}$$

If $\theta \in \left[\theta', \theta^*\right]$, this expression reads

$$\begin{aligned} &(1-\delta)R_E[\theta^* - \theta]\,(E(a(\theta)|\theta \in (\theta_*, \theta^*)) - a^*]) \\ &-(1-\delta)[R_E(1-\theta) - c'(a^*)]\,(E(a(\theta)|\theta \in (\theta_*, \theta^*)) - a^*]) \end{aligned}$$

Since

$$E(a(\theta)|\theta \in (\theta_*, \theta^*)) > a^*,$$

the sign of this expression is the same as the sign of

$$R_E[\theta^* - \theta] - [R_E(1-\theta) - c'(a^*)]$$

Noting that a^* is bounded away from above by $a^{NE}(\theta')$[16],there is a $\varsigma > 0$ so that

$$[R_E(1-\theta) - c'(a_*)] > \varsigma$$

for θ close enough to θ'.

Hence, one can pick θ^* close enough to θ' and make $\frac{d\delta v(\theta;\alpha)}{d\alpha}|_{\alpha=0} < 0$. Hence, those continuation values induce truthtelling when the relevant action schedule is $\widetilde{a}(\theta)$ and are feasible for α close enough to zero. ■

Proof of Proposition 4: In the text.■

Proof of Proposition 5: We proceed in a couple of steps.

STEP 1: It is not optimal to have a differentiable, and strictly decreasing action schedule over $[\underline{\theta}, \overline{\theta}]$.

Proof: Since $v(\theta) = \overline{v}$ for all θ, incentive compatibility in such a case is equivalent to

$$[(1-\theta)R_E - c'(a(\theta))]\frac{da(\theta)}{d(\theta)} = 0.$$

Therefore, the only incentive compatible action schedule which is everywhere differentiable and strictly decreasing over $[\underline{\theta}, \overline{\theta}]$ is $a^{NE}(\theta)$, which is the schedule that satisfies

$$[(1-\theta)R_E - c'(a(\theta))] = 0.$$

Consider replacing $a^{NE}(\theta)$ by

$$a^1(\theta) = \begin{cases} a^{NE}(\theta) \text{ if } \theta \in [\underline{\theta}, \theta^1) \\ a_1 \equiv E\left(a^{NE}(\theta)\,|\theta \in [\theta^1, \overline{\theta}]\right) \text{ otherwise} \end{cases}$$

for some θ^1 close to $\overline{\theta}$. Ignoring, at first, the effect of such a change in type $\overline{\theta}$'s utility, matters will be improved. However, $a^1(\theta)$ is not incentive compatible as $a^1(\theta) < a^{NE}(\theta^1)$ so that, given the continuity of $a^{NE}(\theta)$ in θ, type θ^1 would be better off announcing $\theta^1 - \gamma$ rather than θ^1 for γ small enough. To fix this, pick $\theta_2 < \theta^1$ so that, for $a_2 \equiv E\left(a^{NE}(\theta)\,|\theta \in [\theta^2, \theta^1]\right)$,

$$(1-\theta^1)a_2 R_E - c(a_2) = (1-\theta^1)R_E a_1 - c(a_1).$$

For θ^1 close to $\overline{\theta}$, such a θ^2 exists as (i) $a_2 > a^{NE}(\theta^1) > a_1$, and (ii) $(1-\theta^1)R_E a - c(a)$ is strictly concave in a and attains its highest value at $a^{NE}(\theta^1)$. Replacing $a^1(\theta)$ by

$$a^2(\theta) = \begin{cases} a_2 \text{ if } \theta \in [\theta^2, \theta^1) \\ a_1(\theta) \text{ otherwise} \end{cases}$$

type θ^1 will not have incentives to misreport. Moreover, as for an interval $a^1(\theta)$ was replaced by its conditional expected value, there will be an additional gain the objective. However, $a^2(\theta)$ is not Incentive

[16] Otherwise, a type $\theta = \theta' - \gamma$ would be, for $\gamma > 0$ close enough to zero, strictly better off by announcing θ' as (i) it will allow him to pursue an action that is closer to what is best for him in the stage game (by the continuity of $a^{NE}(\theta)$ in θ), and (ii) grant him the best possible continuation value.

Compatible: type θ^2 will have an incentive to report $\theta^2 - \gamma$ rather than θ^2 for γ small enough. Once again, we can fix this by picking θ^3 so that, for $a_3 \equiv E\left(a^{NE}(\theta)\,|\theta \in [\theta^3, \theta^2)\right)$,

$$(1-\theta^2)R_E a_2 - c(a_2) = (1-\theta^2)R_E a_3 - c(a_3).$$

For the same reasons as above, such a θ^3 exists, the replacement of $a^2(\theta)$ by

$$a^3(\theta) = \begin{cases} a_3 \text{ if } \theta \in [\theta^3, \theta^2) \\ a_2(\theta) \text{ otherwise} \end{cases}$$

precludes a deviation from θ^2, and increases the objective function. Proceeding inductively as above, one will have a sequence of cutoffs $\{\theta_i\}_{i=1}^{N}$ and a corresponding step function $a^N(\theta)$ which is incentive compatible. By noting that, as

$$a^{NE}(\overline{\theta}) = \arg\max_a (1-\overline{\theta})R_E - c(a),$$

if one picks $\{\theta_i\}_{i=1}^{N}$ so that θ^1 is close to $\overline{\theta}$, the resulting loss for type $\overline{\theta}$ coming from the substitution of $a^{NE}(.)$ by $a^N(.)$ will be of second order. As the gains from substituting the action schedule by its mean value are of first order magnitude, the result follows.■

STEP 2: If $a(.)$ is strictly decreasing for some interval (θ_*, θ^*), it cannot jump at either θ_* or θ^*.

Proof: Under the stated assumption, $a(.)$ has to be equal to $a^{NE}(\theta)$ over (θ_*, θ^*).

If $a(.)$ were to jump at, say, θ_*, there would be $\varsigma > 0$ so that

$$a(\theta_*) \geq a^{NE}(\theta_*) + \varsigma.$$

Therefore, by continuity of $a^{NE}(\theta)$, θ_* would be better off announcing $\theta_* + \gamma$ for for γ small enough.■

From Step 2, one sees that, if $a(.)$ is strictly decreasing for some interval (θ_*, θ^*), it must be continuous at θ_*, and θ^*.

STEP 3: For any interval $(\theta_*, \theta^*) \subset [\underline{\theta}, \overline{\theta}]$, an action schedule taking the form of

$$a(\theta) = \begin{cases} a^{NE}(\theta_*) \text{ if } \theta < \theta_* \\ a^{NE}(\theta) \text{ if } \theta \in [\theta_*, \theta^*] \\ a^{NE}(\theta^*) \text{ if } \theta > \theta^* \end{cases}$$

cannot be optimal.

Proof: We can apply exactly the same steps as the ones used to proof Step 1 over the set $[\theta_*, \theta^*]$, and make improvements in the components of the objective function related to the informational rents and the payment to be made by the entrepreneur, D.

Incentive compatibility for types θ^* and θ_* may be violated though. To see why that is the case, consider the case of type θ^*. The procedure calls for replacing $a^{NE}(\theta)$ over $[\theta^1, \theta^*]$ by $a_1 = E\left(a^{NE}(\theta)\,|[\theta^1, \theta^*]\right) > a^{NE}(\theta^*)$.

Two remedies can be adopted. First, one could make $a(\theta) = a_1$ for all $\theta > \theta^*$. Second, one could make $a(\theta) = a'$ for $\theta > \theta^*$, where a' satisfies

$$(1-\theta^*)R_E - c(a_1) = (1-\theta^*)R_E - c(a').$$

If one, starting from $a^{NE}(\theta^*)$, decreases slightly the action for types in $\left[\theta^*, \overline{\theta}\right]$, the effect on the objective is given by[17]

$$-\left(1-\overline{\theta}\right)R_E + c\left(a^{NE}(\theta^*)\right) + R_F\left[1-F(\theta^*)\right]\left[1-E(\theta|\theta>\theta^*)\right] - R_E E\left[\frac{F(\theta)}{f(\theta)}|\theta>\theta^*\right]. \qquad \text{(effect)}$$

If (effect) is positive, by picking θ^1 properly, a gain in the objective can be attained by making $a(\theta) = a'$ for $\theta > \theta^*$. If (Effect) is negative, making $a(\theta) = a_1$ for all $\theta > \theta^*$ generates a gain.

The case of type θ_* can be dealt with in an analogous fashion. ■

Exactly the same steps as the ones in the proof of STEP 3 can be used to show that any IC action schedule which is not a step function cannot be optimal.■

Proof of Proposition 6: Clearly among the PPS that have a constant action being taken and that satisfy the "participation" constraints, the one that has $a(\theta) = a^*(\delta)$ for all θ is the best as the optimum calls for the choice of the smallest action compatible with the constraint associated with observable deviations, which is $a^*(\delta)$. For any non-constant action, one would be be moving even further away from the unconstrained optimum.■

Proof of Proposition 7: Let θ' be type for which $a^*(\delta) = a^{NE}(\theta')$. Consider the following schedule

$$a'(\theta) = \begin{cases} a^*(\delta) \text{ if } \theta \leq \theta' \\ a^{NE}(\theta) \text{ if } \theta > \theta'. \end{cases}$$

This course of action is trivially incentive compatible. Moreover, as for $\theta > \theta'$, $a'(\theta)$ is closer to a than $a^*(\delta)$ it, if feasible, will increase the value of the relationship. To argue that $a'(\theta)$ is feasible, one just needs to show that an observable deviation is not profitable for any type. It trivially holds true for $\theta > \theta'$ as for those types the action prescribed is the best static one. From the definition of $a^*(\delta)$

$$\frac{\delta}{1-\delta}[V(a^*(\delta))) - V^{NE}] = [(1-\underline{\theta})R_E(a^{NE}(\underline{\theta}) - a^*) - (c(a^{NE}(\underline{\theta})) - c(a^*)].$$

Since the value of the relationship under $a'(\theta)$ is larger – for its closer to the unconstrained optimum – than $V(a^*(\delta))$, no type will want to deviate from $a'(\theta)$.

We have shown that $a'(\theta)$ dominates $a^*(\delta)$. We can now use the same arguments as the ones in the proof of Proposition 3, and construct a step function which is incentive compatible that improves upon $a'(\theta)$. The fact that the constraints that prevent observable deviations slack under $a'(\theta)$ will imply that, for a properly chosen step function, they will also slack under the new schedule.■

[17] Note that the two first terms are positive, while the second is negative.

Departamento de Economia PUC-Rio
Pontifícia Universidade Católica do Rio de Janeiro
Rua Marques de Sâo Vicente 225 - Rio de Janeiro 22453-900, RJ
Tel.(21) 35271078 Fax (21) 35271084
www.econ.puc-rio.br
flavia@econ.puc-rio.br