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Abstract

Popular press and some practitioners have warned against threats that buying risky assets pose on agents saving for retirement, children education and other uses. This paper shows that in a standard two-period general equilibrium model where some savers have no risk-sharing motives, there exists a non-negligible set of economies (endowments) and equilibria at which every economic agent is better off if some risky assets are added to riskless securities. Numerical examples actually show that the measure of the set of economies (endowments) with equilibrium allocations associated with trading risky assets that are Pareto superior to when there are only riskless assets can be larger than half the measure of the full set of economies.

Keywords: *general equilibrium; financial innovation; risky assets.*

JEL classification numbers: *D14, D53, E21, E44, G11, G18*

1 Introduction.

The 2008 financial crisis raised an old debate about how bad risky securities can be for families saving for retirement, education of children or other very noble motives. Part of

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the popular press, for example, present the riskiness of securities as a real threat for savers[1]. However, standard economic theory arguments would warn against such deterministically negative view of risky asset, considering that prices may reflect such riskiness, and so whether the mere availability of risky assets to trade in the market may not become such a threat for consumers facing no obvious risk-sharing needs. However, it is not obvious either under which conditions the availability of such risky securities for purchase is "good" for such savers and when it is "bad" (regarding ex-ante welfare). To our knowledge, the related academic literature has not answered this question rigorously.

This paper addresses this last question within the standard one-good, two-period, pure-exchange general equilibrium framework in the tradition of Arrow (1953, 1964)[2]. The model assumes two states of the world in the second period with two types of agents, one whose future-period endowment varies across states (and whose present endowment is lower than the maximum future endowment) and the other receiving a present endowment higher than in the future, but where the future endowment does not vary across states. The intuition is that the former represents a borrower who does face a genuine risk-sharing problem, while the latter represents a saver (or a lender) who only has a genuine intertemporal-consumption smoothing problem, with deterministic (low) future income. There are two assets: one risky and another riskless. As usual in this framework, asset payoffs can vary with the state of nature in the future period. The risky security is the promise of a payment of one unit of the commodity for one state of nature and of a zero payment for the other state. The riskless asset simply pays-off one unit of the good in each state. An important assumption is related to the ability to sell short securities by each consumer. On the one hand, the model assumes that all economic agents can short sell the riskless asset. On the other hand, the model also assumes that only one of the two consumers can sell short the risky asset[3]. This agent, and

[1]For example, on March 24th of 2013, the web site of The Guardian published a report on savers taking more risks searching for greater rewards. As another example, the Yahoo-Finance web site has published on July 16th, 2013 an article called "'Tragedy' for savers pushed to risk, as NO account beats inflation."

[2]As stated by Cass (2006), the modeling approach in this paper follows that of Arrow rather than that in Radner (1972), given that the proofs of existence do not impose "artificial" constraints on the individual portfolios.

[3]One may think these consumers (borrowers) as "company-owners" whose future productivity is uncertain, and so they have risk-sharing needs. To satisfy them they "issue" securities that only pay off when the

only this one, can therefore sell short both assets. The state of nature in which the risky asset pays off 0 is that where the borrower's endowment is low. The consumer facing no risk in his future endowment can only short sell the riskless asset. Nevertheless, the latter agent may find optimal, depending on endowments *and market prices*, to buy the risky asset in addition to trading the riskless one, becoming the lender of the agent selling short the risky security.

The analysis focuses on the comparison between equilibrium allocations for the two cases: one where the risky asset is excluded from trading, the other when it is included for trading. When only the riskless bond is allowed to be traded, equilibrium always exists, given well-known results from the incomplete markets literature. This equilibrium is called a *riskless-security equilibrium.* The latter also states that the corresponding equilibrium allocation is not Pareto efficient (except for very special conditions under endowments and preferences), although it is constrained-Pareto efficient (the "constraints" resulting from the equality of the lender's consumption across both states of nature in the future period).

The introduction of the risky security to the former economy entails the following potential technical problem. Given the lender's short selling constraint on the risky asset it is not obvious that this second economy has necessarily an equilibrium. Such an equilibrium, when it exists, is called a *risky-security equilibrium.* However, the assumptions on individual endowments allow to solve such technical issue. That is, the existence of at least one *risky-security equilibrium* for this second model is guaranteed. By the standard arguments this equilibrium allocation is also clearly always Pareto efficient. After proving existence the analysis turns to the main point of the paper: the comparison of the ex-ante welfare at the risky-security-equilibrium allocation with that in the economy with only the riskless security for given initial endowments. A priori, it is clearly non obvious that for *any* distribution of individual endowments the former implies a Pareto-improvement relative to the latter, due to distributional effects that may be present between the two equilibria. Therefore, the paper proceeds to answer the central question, of whether the introduction of the risky asset can really make *all* agents better off, including lenders who have no risk-sharing motives to purchase such asset. The main result is that there is always a subset with non-empty inte-

productivity is high to the market.

rior (with strictly positive Lebesgue measure) of the individual endowment space (containing the set of *riskless-security equilibrium* allocations) such that, for all economies belonging to that subset, there exists a risky-security equilibrium allocation that is Pareto superior to the corresponding riskless-security equilibrium allocation.[4]

The intuition for this result is more clear when considering an individual endowment point exactly equal to a riskless-security equilibrium allocation. In this special case, in an economy with only a riskless security there is clearly no trade in equilibrium. Also, at such endowment vector only the *expected* intertemporal marginal rate of substitution of the borrower is equal to the intertemporal marginal rate of substitution of the lender. However, the marginal rates of substitution between every pair of states and periods across agents differ at the endowment. Thus, introducing a risky security must improve welfare for both agents. By continuity this intuition also applies for any endowment not too far from any riskless-security equilibrium allocation.

However, the proof also shows that such set can be extended to individual endowment points not too close to such constrained-Pareto efficient allocations. The natural following question then is to know how big such a set can be. The paper provides an answer using a CRRA parameterization for the Bernoulli utility function. The paper provides an analytical characterization of the boundaries of that set. However, the analytical expressions for such boundaries are complex enough to prevent an analytical result concerning the size of the set of individual endowments for which the risky security implies Pareto improvement. This leads to numerical computations showing that the size of such set is bigger when *both* the risk aversion coefficient *and* the probability of the bad state (for the borrower) are larger (but not necessarily when only one of them is larger). This numerical result suggests that financial regulations in countries with simultaneously both higher probability of low income shocks and more risk averse consumers (features that may belong to at least certain Emerging Market economies) should not consider universal bans on trading of risky securities as the ex-ante optimal policy. The result suggests in fact that it is important to first determine the

[4]The property of this component (in the set of endowments) is reminiscent of a property of the smooth Arrow-Debreu model. There, the set of equilibrium allocations—which coincides, courtesy of the two welfare theorems, with the set of Pareto optima—is contained in one component of the set of *regular smooth economies* and equilibrium is unique for endowments in that component (see Balasko, 1975a).

type of risk (i.e., whether the probabilities are well-known or not, due to, e.g., asymmetric information) that may arise in securities markets to understand which types of bad events induce ex-ante inefficiencies and which ones may improve ex-ante efficient risk allocation (even when lenders do not face risky future incomes, as it is the case in the model of this paper).

As stated above, this model essentially uses a framework within the general equilibrium models focusing on financial assets, being them variations on Arrow (1953, 1954)[5]. It is somehow surprising (but, to our knowledge, also true) that none of those well-established contributions in this literature ever reached the main result in this paper concerning the ex-ante welfare properties of introducing risky securities for lenders without risk sharing motives. Thus, one way to view this paper is that it contributes to stress how "standard" general equilibrium theory with risk can provide a sharp answer to such a relevant issue in financial markets.

This paper is clearly related to the literature on financial innovation[6] in general equilibrium. The seminal work of this literature is Hart (1975). This well-known paper shows that it is not always true that the introduction of a new asset in an incomplete financial markets economy improves welfare, due to the possible effects on the equilibrium relative price of physical goods. More recent papers show more explicit conditions under which such an addition in the asset structure implies Pareto improvement or worsening. In particular, in a multiple-commodity model Elul (1995) shows that, on the one hand, there are certain conditions on the number of future states, assets, agents and commodities such that Hart's example of Pareto-inferior financial innovation becomes a generic property, and on the other hand, that under the same conditions there is an asset that implies a Pareto improvement. Thus, it is far from obvious to find a clear direction of the welfare properties of the introduction of a new security to an incomplete markets model.[7] Also, Cass and Citanna (1998)

[5] For a sample of the huge literature devoted to these models, see Balasko and Cass (1989), Cass (1984), Geanakoplos and Mas-Collel (1989), the survey by Magill and Shafer (1991) and the textbook by Magill and Quinzii (1996).

[6] There are quite comprehensive surveys on the literature about financial innovation. See the Allen and Gale (1994) book, the 1995 special issue by the Journal of Economic Theory and also the chapter by Tufano (2003).

[7] In a related work, Chen (1995) shows that frictions in financial markets (such as no short sales constraints)

show a similar result when the degree of market incompleteness in the economy is sufficiently larger than the degree of heterogeneity across consumers. When there is only one commodity, Elul (1999) shows that, generically, it is always possible to find an asset that implies a Pareto improvement[8]. Thus, the main theorem of this paper provides a sharper and more precise result regarding under which conditions financial innovation introducing a risky security really improves ex-ante welfare for all investors, especially to those who have no risk-sharing needs. Indeed, this model not only shows that there is just one asset that improves welfare. What the paper really stresses is that this asset implies no payment in the state in which the short-selling side of the market receive a low income shock. This zero-payment is imposed on lenders who do not face any intrinsic motivation to absorb risk.

This paper is also linked to a recent branch of the credit-transfer literature. A particularly interesting paper is that by Allen and Carletti (2006). That paper uses a numerical example of the financial-intermediation model by Allen and Gale (2004 and 2006) to show that there are conditions for which credit risk transfers between banks and insurance companies may be Pareto improving depending on the distribution of the "liquidity shock" faced by banks[9]. The incentives for risk transfers in that paper come from the asymmetry in the riskiness of investment opportunities faced by each type of intermediary. Clearly, the mechanisms behind those results are quite different to that in this paper, given the differences in the respective economic environments. In particular, the potential mutual benefits from risk transfers in Allen and Carletti's (2006) example comes from a proper risk-sharing improvement. In this paper, instead, agents with future riskless income purchase risky securities that may induce (in equilibrium) to a Pareto improvement.

The rest of the paper is as follows. Section 2 presents the set-up including all possible

may play a key role in financial innovation: this innovation may improve risk sharing's investors.

[8]The literature on financial innovation also includes considerations on incentives to innovate (see, e.g., Allen and Gale (1991) and Pesendorfer (1995)), asymmetric information (see, e.g., Rahi (1995, 1996) and Marin and Rahi (2000)) and competition (Carvajal et al, 2012). This paper clearly abstract from such relevant issues, focusing on the question about the effect of introducing the risky asset in equilibrium over the ex-ante welfare of a well-informed lender.

[9]They also show conditions under which introducing credit risk transfers not may only worse welfare but may also introduce contagion. Clearly that result, though interesting in itself, has no comparison point with the focus on this paper.

securities. Section 3 characterizes the equilibrium first with only the riskless asset and then with the latter, in terms of existence and Pareto efficiency. Section 4 presents the result showing the conditions implying that the addition of the risky security is Pareto-improving. Finally section 5 presents concluding remarks as well as some suggestions for future research.

2 The economy

This section presents the environment, which corresponds to an example of the standard (Arrow's) financial securities model. However, the model in this paper drops the assumption of any economic agent being able to issue (short-sell) any security. Instead, we assume that, for one of the securities, only one agent can sell short units of that security, whereas the other agents can only purchase it.

2.1 The physical environment

There are two time periods and two (types of) consumers, indexed by $i, t = 0, 1$. For reasons to be explained below, call consumer 0 the *borrower* and consumer 1 the *lender*. There is no risk regarding the period-1 endowments of the lender. Risk is then limited to the borrower's period-1 endowment. This allows to introduce the formal equivalent of three "states", namely state 0 (for period 0), and states 1 and 2 depending on the borrower's endowments in period 1. The idea is that states 1 and 2 correspond to low and high levels of resources respectively, as it will be clear below.

For simplicity, suppose that there is only one good in every state. Taking this good as the numeraire, then the spot prices of the good for each state is just equal to 1. At some point, the analysis will consider the standard Arrow-Debreu economy defined by the same 3 goods and the same preferences. Denote by $P \equiv (P(0), P(1), P(2))$ the Walrasian price vector of that Arrow-Debreu (without assets) economy. That price $P = (P(0), P(1), P(2))$ will then be normalized by the condition $P(0) = 1$.

Preferences of consumer $i \in \{0, 1\}$ are represented by a utility function $U_i\left(x_i(0), x_i(1), x_i(2)\right)$ where $x_i = (x_i(0), x_i(1), x_i(2))$ denotes consumer i's consumption of the physical goods. The usual assumption that only strictly positive consumption of physical goods applies here. Let

$X \equiv \mathbb{R}_{++}$ denote the strictly positive orthant of the physical commodity space $\mathbb{R}$, the consumption space of every consumer is X^3. We therefore have that the typical consumption bundle is $x_i = (x_i(0), x_i(1), x_i(2)) \in X^3$. Consumer i's utility function is assumed to satisfy the standard assumptions of smooth consumer theory: 1) smoothness; 2) smooth monotonicity; 3) smooth quasi-concavity; 4) closedness of indifference surfaces in the contingent and dated commodity space $\mathbb{R}$. (See, e.g., Balasko, 1988). As the benchmark case, the model assumes that the utility functions are time and state separable à la Savage. In other words, consumer i's utility can be written as

$$U_i\left(x_i(0), x_i(1), x_i(2)\right) = u_i\left(x_1(0)\right) + \alpha u_i\left(x_i(1)\right) + (1-\alpha) u_i\left(x_i(2)\right) \tag{1}$$

where $\alpha \in (0,1)$ is the probability of state $s = 1$. In further numerical examples the assumption will be

$$u_i\left(x_i\right) = \frac{x_i^{1-\sigma} - 1}{1-\sigma};\ \sigma > 0$$

On the other hand, consumer i is endowed with the commodity bundle $\omega_i = (\omega_i(0), \omega_i(1), \omega_i(2)) \in X^3$. For the borrower (consumer 0) assume that

$$\max\left\{\omega_0\left(0\right), \omega_0(1)\right\} < \omega_0(2) \tag{2}$$

For the lender (agent 1), the assumption is $\omega_1\left(1\right) = \omega_1\left(2\right) \equiv \omega_1^1$ (recalling that there is no risk-sharing intrinsic incentives for the lender). Assume also that

$$\omega_1\left(0\right) > \omega_1^1 \tag{3}$$

In other words, the borrower's endowment in period 1 is uncertain, receiving more goods in state 2 than in both states 0 and 1. This assumption will imply that, in equilibrium, consumer 0 will be interested in borrowing in period 0. On the other hand, the lender faces no risk in his future period's endowment but the latter is lower than lender's period-0 endowment. This assumption will imply that in equilibrium the lender has an intrinsic need for consumption smoothing (this is the reason for imposing the name of lender to consumer 1), in the sense of having incentives to *save* in period 0. Finally, let $r \equiv \left(\omega\left(0\right), \omega\left(1\right), \omega\left(2\right)\right)$ denote the vector of aggregate endowments for this economy.

2.2 Assets

2.2.1 The riskless asset

Every consumer can buy or sell short (issue) a riskless asset. An obvious no arbitrage condition implies that the prices of the riskless assets issued by different economic agents must be proportional. This condition is equivalent to having a unique riskless asset that is issued anonymously. The payoffs of a unit of the riskless asset consist of one unit of numeraire in both states 1 and 2. We denote by q_1 the price (in period 0) of this asset and by b_i^1 its "net demand" by consumer i, with $i \geq 0$. The first case to be analyzed is when the riskless asset is the unique asset.

2.2.2 The risky asset

Alternatively, the model considers the second case where, besides trading in the riskless asset, only the borrower can sell short (issue) a risky security, while the lender can only purchase (non negative amounts of) it. This security's payoff is equal to 0 for state $s = 1$ and to one unit of numeraire for state $s = 2$. Denote by q_0 the price (in period 0) of that risky asset and by b_i^0 the "net demand" of the risky asset by consumer i, with $i \geq 0$. Since this asset is necessarily issued by consumer 0, we must have $b_0^0 \leq 0$ and $b_i^0 \geq 0$ for $i \geq 1$.

3 Benchmark: equilibrium in the economy with the riskless asset only.

This section considers the benchmark case where only the riskless asset is traded. The following section analyzes the economy with both assets, the riskless asset and the risky security. Recall that the riskless asset is issued anonymously and by possibly both the borrower and the lender.

3.1 Individual consumer's problem.

For any agent $i \in \{0,1\}$, consumer i faces one budget constraint for every state of nature $s = 0, 1, 2$. They are as follows:

$$\begin{aligned} x_i(0) - \omega_i(0) &= -q_1 b_i^1 \\ x_i(1) - \omega_i(1) &= b_i^1 \\ x_i(2) - \omega_i(2) &= b_i^1 \end{aligned} \tag{4}$$

Clearly, this consumer's maximization problem has always a solution for any $i \geq 0$ and any price system (p, q_1).

Second, in an equilibrium for the economy defined by the endowment vector $\omega = (\omega_0, \omega_1)$ total supply of the physical goods and assets must equal total demand of them. This gives us the following equilibrium condition:

$$\begin{aligned} \sum_i x_i(s) = \sum_i \omega_i(s),\ s \in \{0,1,2\} \\ \sum_{i \geq 0} b_i^1 = 0. \end{aligned} \tag{5}$$

3.2 Equilibrium analysis

Note that this model is a standard general equilibrium model with one asset with real payoffs. The rank of the payoff matrix is equal to one. It then follows from, e.g., Duffie and Shafer (1985), that equilibrium always exists. In addition, generically on endowments, equilibrium is locally isolated but there may exist multiple equilibria.

Before starting the analysis, it is convenient to introduce a formal tool for the analysis. Let V_i be the plane in the commodity space $\mathbb{R}$ consisting of the points $\omega_i + b_i^1(-q_1, 1, 1)$ where q_1 and b_i^1 are varied in the set of real numbers. It follows from the budget constraints satisfied by consumer i that the consumption bundle $x_i = (x_i(0), x_i(1), x_i(2))$ in the riskless model must belong to the plane V_i. Then, all consumption allocations included in V_1 must satisfy the equality $x_1(1) = x_1(2)$, given the assumption on agent 1's endowments. The equilibrium allocation $x = (x_0, x_1)$ is therefore *constrained Pareto efficient*, the constraint being that consumer i's allocation x_i belongs to the plane V_i. In general, the equilibrium allocations in that riskless model are not Pareto efficient.

3.2.1 A geometric tool: three-dimensional extension of the Edgeworth box

The commodity space is the ordinary three dimensional space $\mathbb{R}^3$ of Solid (i.e., three-dimensional Euclidean) geometry. This enables us to use the three dimensional analog of the Edgeworth box. The minor difficulty due to having an additional dimension is compensated by the intuition brought by the geometric formulation.

As in the standard Edgeworth box, the vector of total endowments $r \in \mathbb{R}^3_{++}$ is fixed. We have two coordinate systems. The first coordinate system is centered at the point O_0 and is used to represent consumer 0's resources, consumption, and preferences. The second coordinate system is centered at O_1, the extremity of the vector of total resources $r \in \mathbb{R}^3$ in consumer 0's coordinate system. The coordinate axes for consumer 1 are parallel and oriented in the opposite direction to those of consumer 0. Let $x = (x_0, x_1)$ be a feasible allocation, i.e., an allocation such that $x_0 + x_1 = r$. Let M be the point in $\mathbb{R}^3$ whose coordinates in consumer 0's coordinate system are x_0. Then, the coordinates of point M in the coordinate system of consumer 1 are then equal to x_1.

3.2.2 Pareto optima and the contract curve in three dimensions

It will become very useful to graphically characterize the Pareto-efficient consumption allocations. To do this, note that the borrower's preferences are represented by a collection of indifference *surfaces* in the borrower's coordinate system. Similarly, the lender's preferences are represented by a collection of indifference surfaces in his own coordinate system. An allocation $y = (y_0, y_1)$ is then *Pareto-efficient* (or, equivalently, it is a *Pareto optimum*) if the point M that represents $y = (y_0, y_1)$ (i.e., the coordinates of M in the borrower's coordinate system are equal to y_0) is the contact point of the indifference surfaces of the two consumers through that point. The budget plane $F(y)$ associated with the Pareto optimum $y = (y_0, y_1)$ is the common tangent plane to the two indifference surfaces that pass through that point. Also, the **supporting Walrasian price vector** $P = (P(0), P(1), P(2))$ for the Pareto optimum $y = (y_0, y_1)$ is perpendicular to the budget plane $F(y)$. The allocation $y = (y_0, y_1)$ is then the **Walrasian equilibrium allocation** associated with the **Walrasian price vector** $P = (P(0), P(1), P(2))$ and any endowment vector $\omega = (\omega_0, \omega_1)$ in that budget plane $F(y)$.

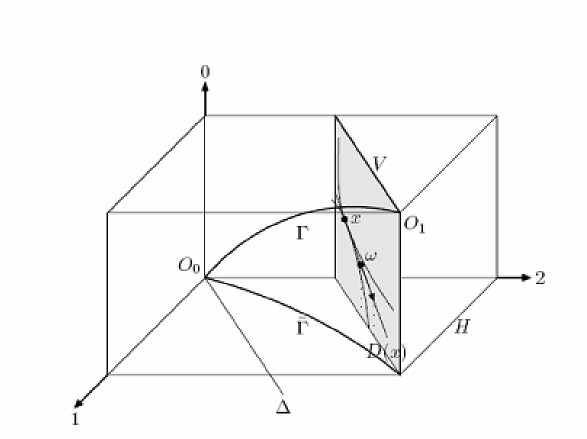

Figure 1. Three-dimensional Edgeworth box
and the plane of individual endowments.

3.2.3 The vertical plane of endowments and of equilibrium allocations with only the riskless asset.

Let the horizontal plane H be defined by the coordinate axes. The plane H represents the goods consumed in states 1 and 2 by the borrower. Define the line Δ as the diagonal of these two axes. Thus, along this diagonal, $x_0(1) = x_0(2)$. Also, subsection 3.2 presents the definition of the planes V_0 and V_1. In the three-dimensional Edgeworth box these two planes become essentially one vertical plane, a plane that contains the point O_1 and that is parallel to the diagonal line Δ. We denote that plane by V. In addition, it follows from the condition $\omega_1(1) = \omega_1(2)$ that reflects the certainty of consumer 1's resources that the endowment vector $\omega = (\omega_0, \omega_1)$ can be any point in the vertical plane V. (The endowment vector $\omega = (\omega_0, \omega_1)$ is actually contained in the box of V defined by the two consumers' coordinate axes if all endowments are to be > 0). Figure 2 below presents a two-dimensional view of plane V.

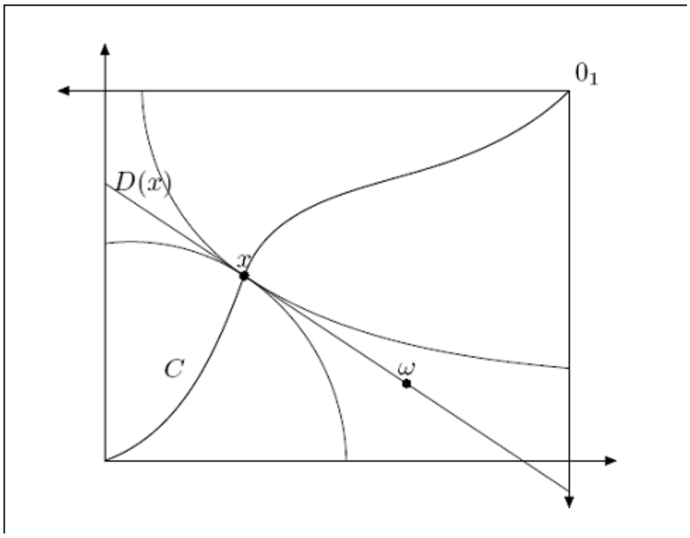

Figure 2. Two-dimensional view of endowments and riskless-security equilibrium plane.

In addition to the endowment vector $\omega = (\omega_0, \omega_1)$ belonging to the vertical plane V, the equilibrium allocations generated by the riskless asset only also belong to the plane V because, since the lender faces no uncertainty on period-1 endowments and can only trade in the riskless security, the obvious result is that this same consumer faces no uncertainty in his period 1 consumption. Therefore, the equilibrium analysis in the model without the risky asset is reduced to the study of the model defined by the restriction of the preferences for both consumers to the plane V. Since endowments and allocations also belong to that plane V, they define a standard two-dimensional Edgeworth box.

The allocation $x = (x_0, x_1)$ is a *riskless-security equilibrium allocation* if and only if x is an equilibrium allocation associated with $\omega = (\omega_0, \omega_1) \in V$ for the two-dimensional Edgeworth box defined by the vertical plane V and the restriction of both consumers' preferences to that plane. Existence of a *riskless-security equilibrium* then follows readily in the current case from the existence of equilibrium for a standard Arrow-Debreu model with two goods and two consumers[10].

It is obvious from figure 1 that the *riskless-security equilibrium allocation* $x = (x_0, x_1)$ is Pareto efficient with respect to all allocations belonging to V. It is therefore a *constrained Pareto optimum.* In general, however, this allocation is not a Pareto optimum for the whole economy (i.e., in the three dimensional space) because both consumers' indifference surfaces

[10]Incidentally, this technique of proof could easily be extended to work for the general case of any number of consumers.

through the equilibrium allocation with only the riskless asset $x = (x_0, x_1)$ do not have the same tangent plane at this point even if the "budget line" defined by the asset without default links the allocation x and the endowment ω and is tangent to both indifference surfaces at $x = (x_0, x_1)$.

Let C be the "contract curve" for the two-dimensional Edgeworth box defined by the vertical plane V and the restrictions of the two consumers' preferences to that plane. The "contract curve" consists of *constrained Pareto optima.* They belong to the plane V and cannot be Pareto dominated by other points of V. Incidentally, varying the plane V (which amounts to varying the distribution of total resources between consumer 0 and 1) generates the surface of *constrained Pareto optima* in the three-dimensional Edgeworth box.

4 The economy with the risky asset

This section reintroduces the risky security in the analysis. As stated below, several differences arise in equilibrium. The graphical tool introduced in the last section will become also a very powerful tool to characterize equilibria when both securities are traded (and where only agent 0 can issue the risky asset).

4.1 Risky-asset equilibrium: main elements.

As above, we introduce the set up for the optimal decision for consumer $i = 1, 2$. In the model with the risky asset (in addition to the riskless one), the budget constraints take the form

$$\begin{aligned} x_i(0) - \omega_i(0) &= -q_0 b_i^0 - q_1 b_i^1 \\ x_i(1) - \omega_i(1) &= b_i^1 \\ x_i(2) - \omega_i(2) &= (b_i^0 + b_i^1) \end{aligned} \tag{6}$$

where, by assumption, $b_0^0 \leq 0$ and $b_1^0 \geq 0$. Consumer i's demand (x_i, b_i) in this case then maximizes his expected utility function $U_i\,(x_i(0), x_i(1), x_i(2))$ subject to the budget constraint (6). The consumer's maximization problem in the model with the risky security has always a solution for any $i \geq 0$ and any price system (p, q_0, q_1).

Therefore, the price system (p, q_0, q_1) is an equilibrium with default of the economy defined by the endowment vector $\omega = (\omega_0, \omega_1)$ if there is equality between total supply and demand of the physical goods and assets, and *if the proper sign constraints are satisfied by the consumers' portfolios.* This gives us the following equilibrium condition:

$$\begin{aligned} \sum_i x_i(s) = \sum_i \omega_i(s),\ s \in \{0,1,2\} \\ \sum_{i\geq 0} b_i^0 = 0, \qquad b_1^0 \geq 0 \text{ for } i \geq 1, \\ \sum_{i\geq 0} b_i^1 = 0. \end{aligned} \tag{7}$$

4.2 Risky-security equilibrium: analysis

The model with both assets is essentially the complete original model presented above, including the individual portfolios sign constraints $b_1^0 \geq 0$. The equilibrium condition $\sum_{i\geq 0} b_i^0 = 0$ then implies the inequality $b_0^0 \leq 0$. Endowment and preferences allow to exclude the trivial case where $b_i^0 = 0$ for all i,where no risky asset is traded in equilibrium. Therefore, from now on, we focus on equilibria with $b_1^0 > 0$. It is when these inequalities are satisfied that we have a *risky-security equilibrium.*

The first stage in the equilibrium analysis is related to existence issues. Indeed, a priori, it may be that no *risky-security equilibrium* exists for a given endowment vector $\omega = (\omega_i)_{i\geq 0}$. In such a case, the economy settles to a riskless-security equilibrium. The issue, therefore, is whether *risky-security equilibrium* can exist and to get a better understanding of when such an equilibrium may exist. Like the case without the risky asset, this analysis benefits from the geometrical tool introduced above.

4.3 Existence of a risky-security equilibrium

We now prove the existence of *risky-security equilibria* for the leading example.

Theorem 1 *Under inequalities (2) and (3), all Walrasian equilibrium allocations associated with any endowment vector $\omega = (\omega_0, \omega_1) \in V$ that is not a Pareto optimum satisfy the portfolio sign conditions that identify them to risky security equilibrium allocations. The latter are Pareto efficient.*

Proof. Before starting with the proper proof, consider the case where the sign constraints $b_1^0 \geq 0$ and $b_0^0 \leq 0$ are ignored. In this case, we have a standard general equilibrium model with two independent assets because the rank of the payoff matrix is equal to two. This model has clearly at least an equilibrium for any endowment vector $\omega = (\omega_i)_{i \geq 0}$.

Note that if we ignore the sign of the portfolios—in other words, we forget provisionally the assumption that the risky asset 0 can be issued only by consumer 0,—the elimination of b_i^0 and b_i^1 between the three budget constraints yields the unique budget constraint that involves only the physical goods:

$$(x_i(0) - \omega_i(0)) + (q_1 - q_0)(x_i(1) - \omega_i(1)) + q_0(x_i(2) - \omega_i(2)) = 0.$$

Therefore, the price vector (q_0, q_1) is a risky-security *equilibrium price vector* (excluding the consideration of the sign constraints on portfolios) if and only if the Walrasian price vector (cf. Section 2.1) $P = (P(0), P(1), P(2))$ where $P(0) = 1$, $P(1) = (q_1 - q_0)$ and $P(2) = q_0$ is an equilibrium price vector of the Arrow-Debreu model defined by the endowment vector $\omega = (\omega_0, \omega_1)$.

Given $\omega = (\omega_0, \omega_1)$ in the vertical plane V, a Walrasian equilibrium exists by the standard existence theorems. The corresponding Walrasian equilibrium allocation $y = (y_0, y_1)$ is a Pareto optimum. At this stage, the question that remains open is whether the portfolio sign constraints that define a risky-security *equilibrium* are satisfied at such Walrasian equilibrium. The remaining part of the proof demonstrates that the answer to the former question is affirmative.

It is clear that if $\omega = (\omega_0, \omega_1) \in V$ is already a Pareto optimum (in the sense of the three-dimensional Edgeworth box), then the corresponding risky-security equilibrium allocation x is uniquely defined and equal to ω, as is the equilibrium allocation y in the three-dimensional Edgeworth box. There is no non trivial risky-security equilibrium allocation.

Let $\omega = (\omega_0, \omega_1) \in V$ that is not a Pareto optimum. Let $y = (y_0, y_1)$ be a Walrasian equilibrium allocation associated with $\omega = (\omega_0, \omega_1) \in V$. This point $y = (y_0, y_1)$ is therefore different from $\omega = (\omega_0, \omega_1)$. Let $F(y)$ be the plane through y that is tangent to the two indifference surfaces that pass through y. Let $P = (P(0), P(1), P(2))$ be the Walrasian price vector perpendicular to the plane $F(y)$ with the normalization $P(0) = 1$. We have $P(0) = 1$,

$P(1) = q_1 - q_0$ and $P(2) = q_0$, from which follows $q_0 = P(1)$ and $q_1 = P(1) + P(2)$. Let $\vec{a}_0$ and $\vec{a}_1$ be the vectors

$$\vec{a}_0 \equiv \begin{bmatrix} -P(1) \\ 0 \\ 1 \end{bmatrix} \qquad \vec{a}_1 \equiv \begin{bmatrix} -P(1) - P(2) \\ 1 \\ 1 \end{bmatrix}$$

These vectors are just the representation of the risky and the riskless assets respectively. The tangent plane $F(y)$ is the plane parallel to the vectors $\vec{a}_0$ and $\vec{a}_1$ through the point $y = (y_0, y_1)$. The endowment vectors $\omega = (\omega_0, \omega_1)$ admits the allocation $y = (y_0, y_1)$ as a *risky-security equilibrium allocation* if it belongs to the half-plane $F_+(y)$ of $F(y)$ that is generated by the vector $b_0^0 \vec{a}_0 + b_0^1 \vec{a}_1$ with $b_0^0 > 0$ while b_0^1 can have any sign.

Let $\bar{y} = (\bar{y}_0, \bar{y}_1)$ denote the orthogonal projection of $y = (y_0, y_1)$ into the horizontal plane H and let $\Delta(\bar{y})$ denote the line parallel to the diagonal Δ through $\bar{y}$. It follows from the assumptions above that $\omega_0(1) < \omega_0(2)$. The projection of the half-plane $F_+(y)$ into the horizontal plane H is therefore the half-plane that contains the point O_1 and that is bounded by the line $\Delta(\bar{y})$.

It also follows from assumptions above that the projection $\bar{y}$ of the Pareto optimum $y = (y_0, y_1)$ is a Pareto optimum for the fictitious economy made of the goods delivered in states 1 and 2. Let $\bar{\Gamma}$ denote the set of Pareto optima for this two-good economy. Again, but now in the horizontal plane H, we have an Edgeworth box. Let Δ' denote the projection of the vertical plane V into H. The line Δ' is the parallel to Δ that passes through the point $\overline{O}_1$, the projection of O_1 in the horizontal plane H.

The tangents to the borrower's indifference curves at points of Δ are perpendicular to the vector $(\alpha, 1 - \alpha)$. Similarly, the tangents to the indifference curves of consumer 1 at points of Δ' are also perpendicular to the vector $(\alpha, 1 - \alpha)$. Combined with the convexity of the two consumers' indifference curves, this implies that the tangency points of the indifference curves belong necessarily to the strip of the plane H determined by the lines Δ and Δ'. This proves that the contract curve $\bar{\Gamma}$, the projection in H of the 3D contract curve Γ is contained in the half-plane delimited by the line Δ' and containing the point O_0. This proves that all points $\omega = (\omega_0, \omega_1) \in V$ belong to the half-plane $F_+(y)$, for any $y \in \Gamma$. This ends the proof that y is therefore a *risky-security equilibrium allocation* for $\omega \in V$. $\blacksquare$

This existence result essentially implies that the portfolio constraints defined by the restriction that only the borrower can issue the risky asset (and so the lender can only purchase it) are not binding at equilibrium. The main reason is the endowment pattern assumed for both agents: the borrower does want to issue a risky security given his endowment in period 1, while the lender wants to save in period 0 by purchasing such risky security.

5 When risky is Pareto superior to riskless in financial markets.

5.1 The main result

The last section shows that, for every endowment vector $\omega = (\omega_0, \omega_1)$ included in the set of all possible endowment vectors V that is not a Pareto optimum, there exists both an equilibrium with only the riskless asset and also an equilibrium with both assets traded (satisfying the inequality constraints on portfolios). The risky-security equilibrium allocation is clearly Pareto efficient while the equilibrium allocation without the risky asset is only constrained-Pareto efficient, but not Pareto efficient. Nevertheless, this welfare difference does not imply that, for given ω, *every* consumer will ex-ante be better off in the risky-security equilibrium because distributional issues may worsen one of the two consumers's welfare at the Pareto optimum. In particular, the lender may clearly be worse off with both assets are traded given that he does not present intrinsic risk-sharing motives for purchasing assets in period 0, but only consumption smoothing needs. Note that even ignoring the portfolio sign constraints it is not true in general that the equilibria of such a model are Pareto superior to those of the first model. In fact, the study of how the equilibria of these two models are related is a special case of the problem of financial innovation. It follows from that literature that those relationships can be quite complex (see the discussion in the introduction).

The next theorem is the main result of the paper, showing that there is always a set of endowments (with non-empty interior) in V such that all agents's ex-ante welfare (for *both* agent 0 and agent 1) increases when going from a riskless-security equilibrium to a risky-security equilibrium for each endowment in that set:

Theorem 2 *Let $D(x)$ be the common tangent at x in C to the indifference curves of the two consumers (whose preferences are restricted to V) through the point x. The subset of the line $D(x)$ that consists of the endowment points such that the riskless-security equilibrium allocation is Pareto dominated by a risky-security equilibrium allocation contains a segment with non-empty interior.*

Proof. Recall that C is the curve that consists of the constrained Pareto optima in the vertical plane V. The curve C is also the set of *riskless-security equilibrium allocations* associated with endowments in the vertical plane V. Note that the line $D(x)$ is the common tangent to the two consumers' indifference surfaces through the point x in the three dimensional space. The line $D(x)$ consists of the endowment vectors $\omega = (\omega_0, \omega_1)$ for which $x = (x_0, x_1)$ is a *riskless-security equilibrium allocation.* We can now characterize the points $\omega = (\omega_0, \omega_1)$ of the line $D(x)$ that, as endowments, have *risky-security equilibrium allocations* that are Pareto superior to the allocation $x = (x_0, x_1)$.

Let Γ be the contract curve consisting of the Pareto optima in the three-dimensional space. Let P_0 (resp. P_1) be the point of Γ with utility for the borrower equal to $u_0(x_0)$ (resp. for consumer 1 equal to $u_1(x_1)$). The points P_0 and P_1 define an arc of the contract curve Γ. This arc consists of the allocations that are three-dimensional Pareto optima and that are Pareto superior to the allocation $x = (x_0, x_1)$.

Let $y = (y_0, y_1)$ be some arbitrary point of the arc P_0P_1 of the curve Γ. The point $y = (y_0, y_1)$ represents an equilibrium allocation (in the standard Walrasian sense) associated with the endowment point $\omega = (\omega_0, \omega_1) \in D(x)$ if and only if $\omega = (\omega_0, \omega_1)$ belongs to the budget plane $F(y)$, associated with the Pareto optimum $y = (y_0, y_1)$.

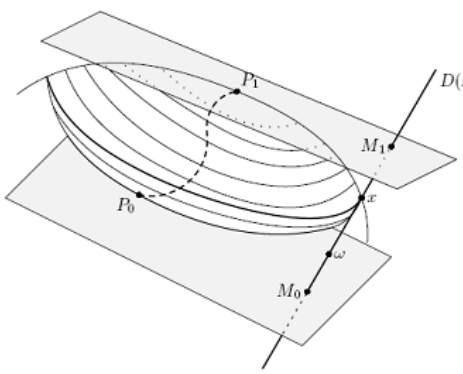

Figure 3. Pareto improvement with risky securities.

Consider the intersection of the plane $F(y)$ with the line $D(x)$. There are two possibilities: (1) for some y, the plane $F(y)$ contains the line $D(x)$; (2) for all y belonging to the arc P_0P_1, the plane $F(y)$ either intersects the line $D(x)$ at just a point or is parallel to $D(x)$.

In the first case, for any endowment vector $\omega = (\omega_0, \omega_1)$ that belongs to $D(x)$, the riskless-security equilibrium allocation $x = (x_0, x_1)$ is Pareto dominated by $y = (y_0, y_1)$. The allocation $y = (y_0, y_1)$ is a *risky-security equilibrium allocation* for **all the points** of the line $D(x)$.

In the second case, let us compactify the line $D(x)$ by adding a point at "infinity." The map $y \to \{D(x) \cup \{\infty\}\} \cap F(y)$ is then defined on the arc P_0P_1 and is continuous. Its image of the connected arc P_0P_1 is a connected subset of $\{D(x) \cup \{\infty\}\}$, a set that is homeomorphic to the circle S^1. The connected subsets of S^1 are the intervals. Therefore, the image of this map is an interval of the line compactified by a point at infinity $\{D(x) \cup \{\infty\}\}$. This interval contains the points M_0 and M_1 that are the intersection of the budget planes $F(P_0)$ and $F(P_1)$ with the line $\{D(x) \cup \{\infty\}\}$ and also contains the point $x = (x_0, x_1)$. In addition, it follows from the convexity and orientation of the indifference surfaces of the two consumers through the point $x = (x_0, x_1)$ that x belongs to the segment M_0M_1.

It then follows from the connectedness property of the image that all the endowments that belong to the segment M_0M_1 are such that there exists an risky-security equilibrium allocation that is Pareto superior to the riskless-security equilibrium allocation $x = (x_0, x_1)$. ■

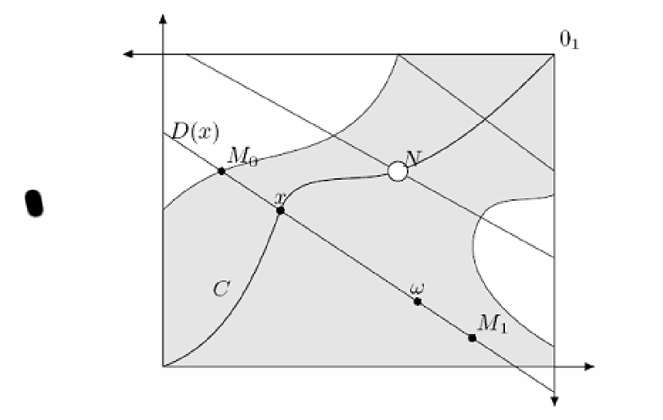

Figure 4. Economies where risky-security equilibria Pareto dominates riskless-security ones.

The economic meaning of this result is clear. Endowments in this segment correspond to those "close to" no-risky-asset equilibrium allocations. This means that, if the endowments for both the borrower and the lender are such that when only the riskless asset is traded, the volume of trading is sufficiently low (close to 0), then this means that opening a new market where the risky security is traded improves ex-ante welfare, not only to the borrower", who has intrinsic risk sharing needs, but also to the lender, who only wants to smooth consumption over time, except in the degenerate case where $x = (x_0, x_1)$ is already a Pareto optimum instead of just being a constrained Pareto optimum. Endowments corresponding to exact riskless-asset equilibrium allocations satisfies the equality between the intertemporal marginal rate of substitution (between consumption of periods 0 and 1) for the lender and the *expected* intertemporal marginal rate of substitution for the borrower. When endowments satisfy conditions not too far from this one then one expects that this introducing this risky asset implies a Pareto improvement.

Indeed, the theorem above shows that those endowments may be "not close enough" to those corresponding to riskless-security equilibrium allocations. The graphical argument shows that the size of the segment of $D(x)$ containing such endowments may depend on the curvature of the indifference surfaces of agents 0 and 1. The size of the set of endowments where Pareto improvement holds can be seen in Figure 4. Note that the curve C contains at least one three-dimensional Pareto optimum obtained by intersecting the three-dimensional contract curve Γ with the vertical plane V. Let N be such a Pareto optimum. When the point x of C tends to N, then the line $D(x)$ tends to a limit position which is the intersection of the budget plane that supports the Pareto optimum N and the vertical plane V. In addition, the intersection points M_0 and M_1 tend also to limit positions. The latter are generally different from the point N.

By varying the constrained Pareto optimum $x = (x_0, x_1)$ along the contract curve C, the interval defined in Proposition 2 generates the set of endowment points in the vertical plane V for which there exists a risky-security equilibrium allocation that Pareto dominates the riskless-security equilibrium allocation. Provided that we include in that definition the case where the riskless-security equilibrium allocation is already Pareto efficient (in which case it is trivially improved by the introduction of the risky asset), then we see that the set of economies where the introduction of the risky security is a Pareto improvement over the no risky asset case contains the curve C. In addition, this set has a non empty interior and contains the subset bounded by the curves generated by the points M_0 and M_1.

5.2 How "likely" is that risky securities lead to Pareto improvement? The case of CRRA preferences..

Given the general result in theorem 2, an important question that arises is, for given aggregate endowments and preferences for both agents, how *big* is the size of individual endowments for which both borrower and lender are better off with the risky security than without it. Given that this Pareto improvement result is not necessarily universally valid (although the proposition shows that the Lebesgue measure of the above mentioned set is strictly positive) then it is clear that, the bigger is the size of such set, the more general is this feature.

Both figure 3 and the proof of theorem 2 indicate the steps to characterize such a set. For every possible value of the expected utility of one of the consumers (say, the borrower), compute the allocation corresponding to the constrained-Pareto efficient allocation consistent to that utility (point x in figure 3), and also compute the expected utility reached by the other agent (the lender) at that allocation. Use the two values of the expected utility to obtain two Pareto-efficient allocations, each one corresponding to each utility value (points P_0 and P_1 in figure 3). After computing the gradient in each of the two allocations, use the gradients two project the two allocations onto the plane that contains all possible individual endowments (i.e., endowments such that $\omega_1(1) = \omega_1(2)$) to get the allocations such as M_1 and M_0. By changing the values of the utility originally chosen (from the minimum value to the maximum possible value) the set can be generated in this way.

A general characterization is not possible, so the remaining of this subsection presents a family of utility functions corresponding to the CRRA case (constant relative risk aversion), i.e.:

$$u_i(x_i) = \begin{cases} \frac{x_i^{1-\sigma}-1}{1-\sigma} \text{ if } \sigma \neq 1, \ \sigma > 0 \\ \ln x_i \text{ if } \sigma = 1 \end{cases} \tag{8}$$

The appendix shows the proof of the following analytical characterization of such a set for CRRA preferences.

Proposition 3 *Fix $\bar{U}_0 \in \left(\bar{U}_{0,\min}, \bar{U}_{0,\max}\right)$, where*

$$\bar{U}_{0,\min} \equiv \begin{cases} \frac{(1-\alpha)\left(\omega^1(2)-\omega^1(1)\right)^{1-\sigma}-2}{1-\sigma} \text{ if } \sigma < 1 \\ -\infty \text{ if } \sigma \geq 1 \end{cases}$$

$$\bar{U}_{0,\max} \equiv \frac{\left(\omega^0\right)^{1-\sigma}-1}{1-\sigma} + \alpha\frac{\left(\omega^1(1)\right)^{1-\sigma}-1}{1-\sigma} + (1-\alpha)\frac{\left(\omega^1(2)\right)^{1-\sigma}-1}{1-\sigma}$$

Let x_1^1 be implicitly defined by

$$\begin{aligned} &(1-\sigma)\bar{U}^0 + 2 \\ &= \left[\frac{\omega^0}{1+x_1^1\left[\sum_{s=1}^2 \alpha(s)\left(\frac{1}{\omega^1(s)-x_1^1}\right)^{\sigma}\right]}\right]^{1-\sigma} + \sum_{s=1}^2 \alpha(s)\left(\omega^1(s)-x_1^1\right)^{1-\sigma} \end{aligned} \tag{9}$$

let $\bar{U}_1\left(\bar{U}_0\right)$ be given by

$$\begin{aligned} \bar{U}_1\left(\bar{U}^0\right) = &\frac{\left[\left\{\frac{x_1^1\left(\bar{U}^0\right)\left[\alpha\left(\omega^1(1)-x_1^1\left(\bar{U}^0\right)\right)^{-\sigma}+(1-\alpha)\left(\omega^1(2)-x_1^1\left(\bar{U}_0\right)\right)^{-\sigma}\right]}{1+x_1^1\left(\bar{U}^0\right)\left[\alpha\left(\omega^1(1)-x_1^1\left(\bar{U}^0\right)\right)^{-\sigma}+(1-\alpha)\left(\omega^1(2)-x_1^1\left(\bar{U}_0\right)\right)^{-\sigma}\right]}\right\}\omega^0\right]^{1-\sigma}-1}{1-\sigma} \\ &+ \frac{\left(x_1^1\left(\bar{U}^0\right)\right)^{1-\sigma}-1}{1-\sigma} \end{aligned} \tag{10}$$

and let

$$\widehat{x}_0^0\left(\bar{U}_0\right) \equiv \omega^0\left\{\frac{(1-\sigma)\bar{U}^0+2}{\left[\left(\omega^0\right)^{1-\sigma}+\alpha\left(\omega^1(1)\right)^{1-\sigma}+(1-\alpha)\left(\omega^1(2)\right)^{1-\sigma}\right]}\right\}^{\frac{1}{1-\sigma}}$$

$$\widehat{x}_0^0\left(\bar{U}_1\right) \equiv \omega^0\left\{\frac{(1-\sigma)\bar{U}_1+2}{\left[\left(\omega^0\right)^{1-\sigma}+\alpha\left(\omega^1(1)\right)^{1-\sigma}+(1-\alpha)\left(\omega^1(2)\right)^{1-\sigma}\right]}\right\}^{\frac{1}{1-\sigma}}$$

Then the upper and lower bounds of the set containing the risky-asset-Pareto-improving endowments are (for $i = 0, 1$ respectively):

$$\bar{\omega}_1^1\left(\bar{U}_i\right) = \frac{2\left(\omega^0-\widehat{x}_0^0\left(\bar{U}_i\right)\right) - \frac{2x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]\omega^0}{1+x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]}}{\left(\omega^0\sum_{s=1}^2\frac{\alpha(s)}{\omega^1(s)}\right) - \left[\frac{\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]\omega^0}{1+x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]}\right]} \tag{11}$$

$$\bar{\omega}_1^0\left(\bar{U}_i\right) = \frac{\left[\frac{2x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]}{1+x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]}\right]\left[\omega^0\left(\sum_{s=1}^2\frac{\alpha(s)}{\omega^1(s)}\right)-2\left(\omega^0-\widehat{x}_0^0\left(\bar{U}_i\right)\right)\right]}{\left(\sum_{s=1}^2\frac{\alpha(s)}{\omega^1(s)}\right) - \left[\frac{\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]}{1+x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]}\right]} \tag{12}$$

The last proposition shows that, even for CRRA preferences (widely used in macroeconomic models) the characterization of the size of that set yields expressions which are complicated to interpret. In particular, it is virtually impossible to provide an analytical result from expressions (11) and (12) regarding how large is this set relative to the whole plane V containing all possible individual endowments. Thus, to answer the question on the size of the set of Pareto-improving endowments makes necessary at least the parametrization of the utility function to get a numerical characterization of the equilibria.

Thus, for illustration, the values for the aggregate endowments are arbitrarily chosen in the following values: $\omega(0) = 4$, $\omega(1) = 1$, $\omega(2) = 3$.We numerically compute the boundaries for the set of individual endowments for which the introduction of a risky asset implies Pareto improvement in the context of the CRRA preferences case. The following figure shows this set when σ is equal to 0.5 and α (the probability of state 1, in which default would occur if a risky security is traded) is equal to 0.2.

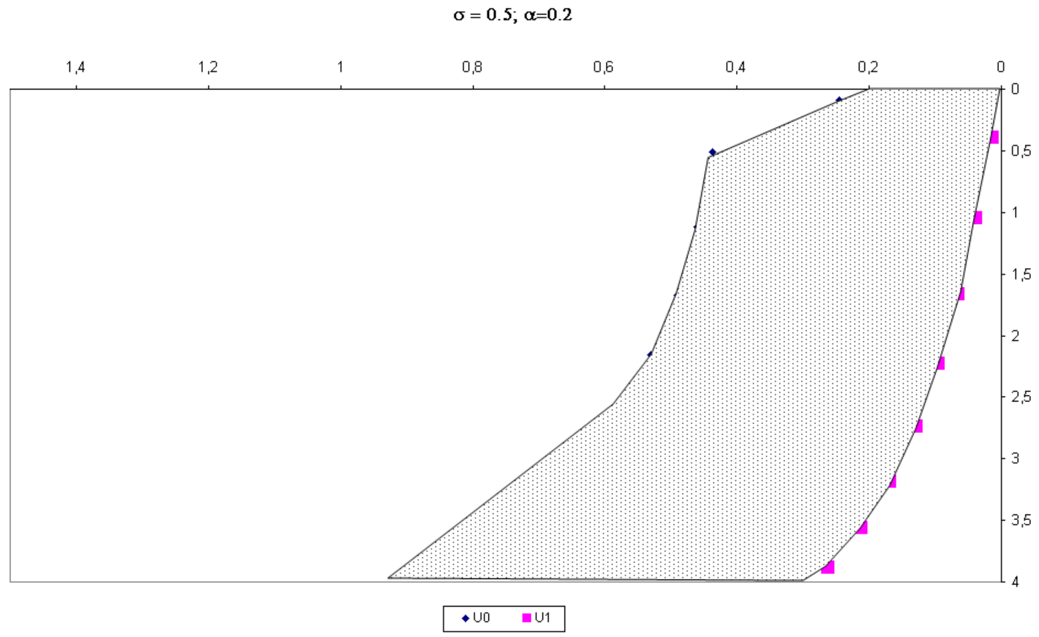

Figure 5: set of individual endowments such that trading in the risky asset Pareto improves ex-ante welfare when $\alpha = 0.2$ and $\sigma = 0.5$.

The whole rectangle in figure 5 represents the set V_1 in this numerical example. The set of endowments in V_1 such that trading in the risky asset implies Pareto improvement is included in the shaded-dotted area in V_1. The message of this figure is that, at least for the proposed parameter values, the size (Lebesgue measure) of the set of individual endowments for which Pareto improvement occurs is far from being negligible. However, the second natural question is whether the endowments on the boundary of such a set depend on parameters such as risk aversion or the probability of default. In fact, the characterization of the set coming from the proposition 2 and figure 5.1 suggests that the size of the set of risky-security-Pareto-improving endowments depend on the curvature of the indifference surfaces, and the structure of the model suggests that such a curvature is related to either the risk aversion or the probability of state 1 (the "bad" state).

It turns out that, for the value of the probability of state 1 equal to 0.2, higher values of σ (the relative-risk-aversion coefficient) does not necessarily increase the size of such a set. Computations[11] done for values of σ equal to 1.1 and 2 show that the set of such endowments (on the plane V_1) *shifts* downwards to the right relative to the benchmark case shown in figure **??**. Also, computations obtained for $\sigma = 0.25$ imply a set shifted upwards and to the left relative to the the set in figure **??**.

We also computed such a set for the case of $\alpha = 0.4$ instead of 0.2 (maintaining the value of σ in 0.5). This exercise essentially tries to link the set of endowments for which trading in the risky security implies a Pareto improvement with the probability of state 1. However, such increase in the probability of the bad state does not increase the size of the set either. Instead, it shifts it to the left and upwards.

However, when considering *simultaneously both* a higher value of the risk aversion coefficient *and* a higher value of the probability of the bad state the size of the set undoubtedly increases. The following figure shows the set for the same values of aggregate endowments

[11] For brevity the figures are not shown here. Such figures are available upon request.

and when the coefficient of relative risk aversion is 0.75 and the probability of state 1 is 0.4.

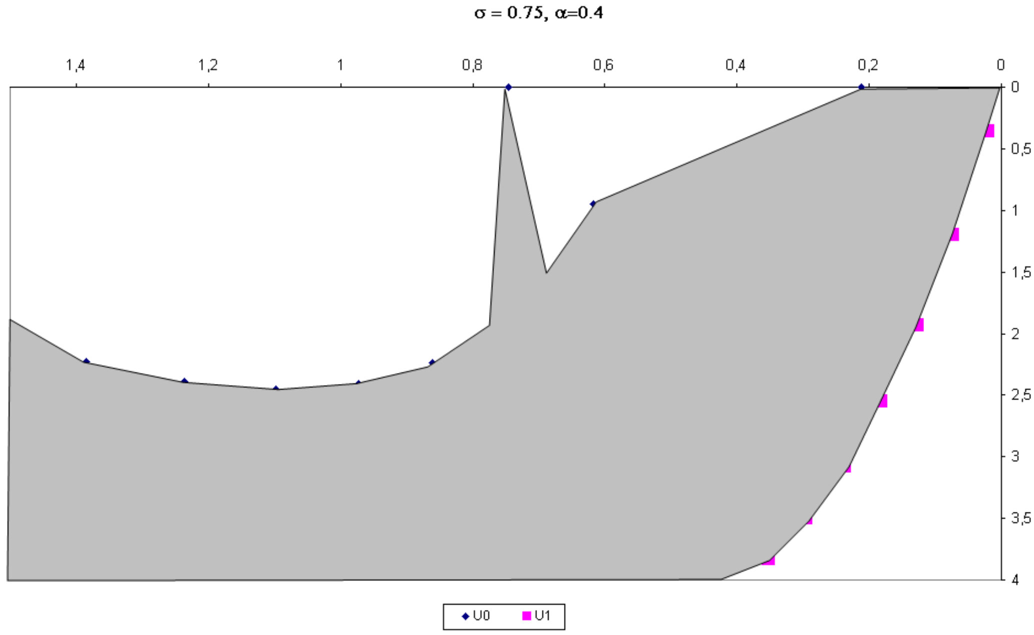

Figure 6: set of individual endowments such that the risky asset Pareto improves ex-ante welfare when $\alpha = 0.4$ and $\sigma = 0.75$.

In figure 6 the set of endowments for which default improves ex-ante efficiency is given by the dark-grey shaded area. Provided that the size of the plane V_1 is the same in both figures (**??**) and (**??**), just by inspection of the boundaries of both sets it is obvious that the green surface in figure **??** is larger than the light-blue one in figure **??**.

This feature is also seen in the following figure, which considers the same value of α (0.4) and a value of risk aversion equal to 0.9.

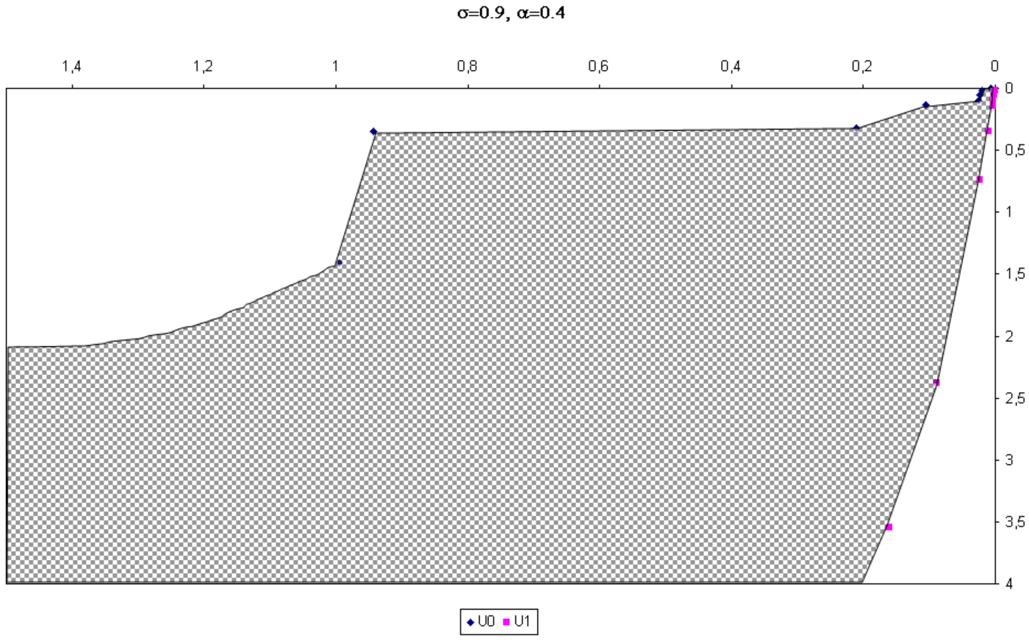

Figure 7: set of individual endowments such that trading in the risky asset Pareto improves ex-ante welfare when $\alpha = 0.2$ and $\sigma = 0.9$.

In figure 7 the set of endowments for which default Pareto improves welfare is the squared-grey shaded area, whose size is even larger than the green area in figure **??**. Note that in this case the shaded surface occupies more than half of the surface of V_1.

Interestingly, one interpretation of the last numerical results would be that introducing risky securities is more *likely* to be Pareto improving in economies where the likelihood of the bad state is higher and with more risk averse agents (assuming symmetric preferences). Being less rigorous with the interpretation, such results suggest that more "fragile" economies (in the sense of being more prone to adverse shocks in future output and being populated with more risk-averse investors) should be more prone to issue risky securities than less "fragile" economies, as long as this risk does not arise from factors ignored in this paper such as asymmetric information or credibility issues, factors that may induce the arising of a very different type of risk (more linked to, e.g., "fraud"). In other words, the results in this section indicates that financial regulators in such "fragile" economies, in their efforts of protecting lenders, and also if guided by ex-ante Pareto-efficiency criterion, do not have enough justification to ban the trading of risky securities, but only justification of risks that arise due to "strategic" decisions by debtors. Of course, this paper does not intend to provide a definitive answer on financial regulation concerning risky securities. However, it does give a warning concerning possible policy-proposals that ban the trading of such securities, without seeing the possible ex-ante benefits even for lenders facing no risk sharing needs.

6 Conclusion

This paper emphasizes that, as it is, an extremely basic version of the standard general equilibrium analysis with incomplete markets developed after Arrow (1953,1964) can answer the question of when the trading in the risky security improves ex-ante welfare even for lenders who do not face intrinsic risk sharing needs. The main theorem of this paper states that when individual endowments are around consumption allocations such that the lender's intertemporal marginal rate of substitution equal the borrower's expected intertemporal marginal rate of substitutions (when borrowers do face uncertainty in their future income) then the risky asset implies a Pareto improvement. Also, numerical computations using CRRA preferences show that the set of economies where this risky-security Pareto-improvement property holds is not negligible and that it is larger the more risk averse and the higher is the probability of the bad state. This conclusion seems to have been largely ignored in the macroeconomics literature, especially in the light of important policy discussions regarding regulation of trading on derivative and other related financial markets.

Clearly, this model uses an important number of simplifying assumptions, such as a unique physical commodity and only two types of agents. For example, as stated in the introduction, Hart (1975) found with more than one commodity that the introduction of a new asset may imply a Pareto worsening rather than an improvement, due to relative price effects. Therefore, it is important to see whether the same type of argument can be applied to a world with many goods. Similarly, a higher degree of heterogeneity may imply a more subtle characterization of endowments where the Pareto improvement result holds. The extension to more general models involving an arbitrary number of goods and consumers should clearly be the subject of further research.

7 Appendix

7.1 Proof of proposition 3.

This subsection follows the logic of theorem (?) to fully characterize the private endowments for which introducing the risky security improves welfare for all agents in the CRRA case. The first step is to characterize the constrained efficient allocations, i.e., those arising in an equilibrium with the riskless asset only. The five equations that characterize these allocations are as follows:

$$\bar{U}^0 = \frac{\left(x_0^0\right)^{1-\sigma} - 1}{1-\sigma} + \alpha \left(\frac{\left(x_0^1(1)\right)^{1-\sigma} - 1}{1-\sigma} \right) + (1-\alpha) \left(\frac{\left(x_0^1(2)\right)^{1-\sigma} - 1}{1-\sigma} \right)$$

$$\left(\frac{x_1^0}{x_1^1} \right)^{\sigma} = \alpha \left(\frac{x_0^0}{x_0^1(1)} \right)^{\sigma} + (1-\alpha) \left(\frac{x_0^0}{x_0^1(2)} \right)^{\sigma} \tag{13}$$

$$\sum_{i=0}^{1} x_i^0 = \omega^0;\; x_0^1(s) + x_1^1 = \omega^1(s),\; s \in \{1,2\} \tag{14}$$

for given $\bar{U}^0$. Replacing the last three feasiblity equations in the second one:

$$x_1^0 = \left(\omega^0 - x_1^0\right) x_1^1 \left[\alpha \left(\frac{1}{\omega^1(1) - x_1^1} \right)^{\sigma} + (1-\alpha) \left(\frac{1}{\omega^1(2) - x_1^1} \right)^{\sigma} \right]^{\frac{1}{\sigma}}$$

so

$$x_1^0 = \left\{ \frac{x_1^1 \left[\alpha \left(\frac{1}{\omega^1(1)-x_1^1} \right)^{\sigma} + (1-\alpha) \left(\frac{1}{\omega^1(2)-x_1^1} \right)^{\sigma} \right]}{1 + x_1^1 \left[\alpha \left(\frac{1}{\omega^1(1)-x_1^1} \right)^{\sigma} + (1-\alpha) \left(\frac{1}{\omega^1(2)-x_1^1} \right)^{\sigma} \right]} \right\} \omega^0 \tag{15}$$

and therefore, using feasibility again for $t = 0$

$$x_0^0 = \frac{\omega^0}{1 + x_1^1 \left[\alpha \left(\frac{1}{\omega^1(1)-x_1^1} \right)^{\sigma} + (1-\alpha) \left(\frac{1}{\omega^1(2)-x_1^1} \right)^{\sigma} \right]}$$

Thus, replacing the latter and period-1 feasibility conditions into the first equation

$$\begin{aligned} &(1-\sigma)\,\bar{U}^0 + 2 \\ &= \left[\frac{\omega^0}{1 + x_1^1 \left[\sum_{s=1}^{2} \alpha(s) \left(\frac{1}{\omega^1(s)-x_1^1} \right)^{\sigma} \right]} \right]^{1-\sigma} + \sum_{s=1}^{2} \alpha(s) \left(\omega^1(s) - x_1^1 \right)^{1-\sigma} \end{aligned} \tag{16}$$

This implicitly determines x_1^1 as a function of $\bar{U}^0$.

Also, the last steps allow to get the level of utility that agent 1 gets in this constrained efficient allocation:

$$\bar{U}_1 = \frac{\left[\left\{\frac{x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]}{1+x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]}\right\}\omega^0\right]^{1-\sigma} - 1}{1-\sigma} + \frac{\left(x_1^1\right)^{1-\sigma}-1}{1-\sigma} \tag{17}$$

Next is the set of equations characterizing the ex-ante (unconstrained) Pareto efficient allocations:

$$\frac{\widehat{x}_0^0}{\widehat{x}_1^0} = \frac{\widehat{x}_0^1(s)}{\widehat{x}_1^1(s)},\ s \in \{1,2\},\ \alpha(s) = \begin{cases} \alpha \text{ if } s \in 1 \\ 1-\alpha \text{ if } s \in 2 \end{cases}$$

$$\sum_{i=0}^{1} \widehat{x}_i^0 = \omega^0;\ \sum_{i=0}^{1} \widehat{x}_i^1(s) = \omega^1(s),\ s \in \{1,2\}$$

According to the proof of theorem (?) the last equation to close the system depends on whether $\bar{U}^0$ or $\bar{U}^1$ is taken as parametric (as the constraint). In the first case, the last equation is

$$\bar{U}^0 = \frac{\left(\widehat{x}_0^0\right)^{1-\sigma}}{1-\sigma} + \alpha\frac{\left(\widehat{x}_0^1(1)\right)^{1-\sigma}}{1-\sigma} + (1-\alpha)\frac{\left(\widehat{x}_0^1(2)\right)^{1-\sigma}}{1-\sigma} - \frac{2}{1-\sigma}$$

Thus, in this case the first part of the system becomes:

$$\left(\frac{\widehat{x}_0^0}{\omega^0 - \widehat{x}_0^0}\right) = \left(\frac{\widehat{x}_0^1(1)}{\omega^1(1) - \widehat{x}_0^1(1)}\right) = \left(\frac{\widehat{x}_0^1(2)}{\omega^1(2) - \widehat{x}_0^1(2)}\right)$$

so

$$\widehat{x}_0^0\left(\omega^1(1) - \widehat{x}_0^1(1)\right) = \widehat{x}_0^1(1)\left(\omega^0 - \widehat{x}_0^0\right)$$

$\Leftrightarrow$

$$\widehat{x}_0^1(1) = \frac{\omega^1(1)\,\widehat{x}_0^0}{\omega^0}$$
$$\widehat{x}_0^1(2) = \frac{\omega^1(2)\,\widehat{x}_0^0}{\omega^0}$$

So in this case

$$\begin{aligned}&(1-\sigma)\bar{U}^0 + 2 \\ &= \left(\widehat{x}_0^0\right)^{1-\sigma}\left[1 + \alpha\left(\frac{\omega^1(1)}{\omega^0}\right)^{1-\sigma} + (1-\alpha)\left(\frac{\omega^1(2)}{\omega^0}\right)^{1-\sigma}\right]\end{aligned} \tag{18}$$

This determines $\widehat{x}_0^0\left(\bar{U}^0\right)$:

$$\widehat{x}_0^0\left(\bar{U}^0\right)=\omega^0\left\{\frac{(1-\sigma)\,\bar{U}^0+2}{\left[\left(\omega^0\right)^{1-\sigma}+\alpha\left(\omega^1\left(1\right)\right)^{1-\sigma}+\left(1-\alpha\right)\left(\omega^1\left(2\right)\right)^{1-\sigma}\right]}\right\}^{\frac{1}{1-\sigma}} \tag{19}$$

From here we can also get

$$\widehat{x}_0^1\left(s,\bar{U}^0\right)=\omega^1\left(s\right)\left\{\frac{(1-\sigma)\,\bar{U}^0+2}{\left[\left(\omega^0\right)^{1-\sigma}+\alpha\left(\omega^1\left(1\right)\right)^{1-\sigma}+\left(1-\alpha\right)\left(\omega^1\left(2\right)\right)^{1-\sigma}\right]}\right\}^{\frac{1}{1-\sigma}} \tag{20}$$

In the second case, the same can be done for the allocations for given $\bar{U}^1$:

$$\widehat{x}_0^0\left(\bar{U}^1\right)=\omega^0\left\{\frac{(1-\sigma)\,\bar{U}^1+2}{\left[\left(\omega^0\right)^{1-\sigma}+\alpha\left(\omega^1\left(1\right)\right)^{1-\sigma}+\left(1-\alpha\right)\left(\omega^1\left(2\right)\right)^{1-\sigma}\right]}\right\}^{\frac{1}{1-\sigma}}$$

$$\widehat{x}_0^1\left(s,\bar{U}^1\right)=\omega^1\left(s\right)\left\{\frac{(1-\sigma)\,\bar{U}^1+2}{\left[\left(\omega^0\right)^{1-\sigma}+\alpha\left(\omega^1\left(1\right)\right)^{1-\sigma}+\left(1-\alpha\right)\left(\omega^1\left(2\right)\right)^{1-\sigma}\right]}\right\}^{\frac{1}{1-\sigma}}$$

where $\bar{U}^1$ is given by equation (17).

Note that, given the symmetry of CRRA preferences the relative price of Arrow-Debreu commodities evaluated at the Pareto-efficient allocation is simply (for any $i=1,\,2$):

$$\widehat{p}\left(s\right)=\frac{\alpha\left(s\right)\widehat{x}_i^0}{\widehat{x}_i^1\left(s\right)}=\frac{\alpha\left(s\right)\omega^0}{\omega^1\left(s\right)};\ s\in\{1,2\} \tag{21}$$

for any value of utility $\bar{U}_i$

The next step is to write the budget constraint equations that characterize the limit point of the sets. On the one hand, each limit point $\left(\bar{\omega}_0^0;\left(\bar{\omega}_0^1\left(s\right)\right)_{s=1}^2\right)$ must satisfy the budget constraint:

$$\bar{\omega}_0^0+\sum_{s=1}^{2}\widehat{p}\left(s\right)\bar{\omega}_0^1\left(s;\bar{U}_i\right)=\widehat{x}_0^0\left(\bar{U}_i\right)+\sum_{s=1}^{2}\widehat{p}\left(s\right)\widehat{x}_0^1\left(s;\bar{U}_i\right)$$

or, using feasibility:

$$\omega^0+\sum_{s=1}^{2}\widehat{p}\left(s\right)\omega^1\left(s\right)-\bar{\omega}_1^0-\bar{\omega}_1^1\left(\sum_{s=1}^{2}\widehat{p}\left(s\right)\right)=\widehat{x}_0^0\left(\bar{U}_i\right)+\sum_{s=1}^{2}\widehat{p}\left(s\right)\widehat{x}_0^1\left(s;\bar{U}_i\right)=2\widehat{x}_0^0\left(\bar{U}_i\right)$$

or

$$\bar{\omega}_1^0+\left(\omega^0\sum_{s=1}^{2}\frac{\alpha\left(s\right)}{\omega^1\left(s\right)}\right)\bar{\omega}_1^1=2\left(\omega^0-\widehat{x}_0^0\left(\bar{U}_i\right)\right) \tag{22}$$

The second equation uses a different relative price:

$$\widetilde{p}=\frac{x_1^0}{x_1^1}=\frac{\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]\omega^0}{1+x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]}$$

where x_1^0 comes from (15) and x_1^1 satisfies (16). Thus, the second equation must satisfy:

$$\bar{\omega}_0^0+\widetilde{p}\bar{\omega}_1^1=x_1^0+\widetilde{p}x_1^1=\frac{2x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]\omega^0}{1+x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]}$$

or

$$\begin{aligned}&\bar{\omega}_1^0+\left[\frac{\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]\omega^0}{1+x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]}\right]\bar{\omega}_1^1\\&=\frac{2x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]\omega^0}{1+x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]}\end{aligned}\tag{23}$$

Thus, solving for $(\bar{\omega}_1^0, \bar{\omega}_1^1)$ as a function of $\bar{U}_i$ from equations (22) and (23):

$$\bar{\omega}_1^1\left(\bar{U}_i\right)=\frac{2\left(\omega^0-\widehat{x}_0^0\left(\bar{U}_i\right)\right)-\frac{2x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]\omega^0}{1+x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]}}{\left(\omega^0\sum_{s=1}^2\frac{\alpha(s)}{\omega^1(s)}\right)-\left[\frac{\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]\omega^0}{1+x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]}\right]}\tag{24}$$

$$\begin{aligned}\bar{\omega}_1^0&=2\left(\omega^0-\widehat{x}_0^0\left(\bar{U}_i\right)\right)\\&-\frac{\left[2\left(\omega^0-\widehat{x}_0^0\left(\bar{U}_i\right)\right)-\frac{2x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]\omega^0}{1+x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]}\right]\left(\sum_{s=1}^2\frac{\alpha(s)}{\omega^1(s)}\right)}{\left(\sum_{s=1}^2\frac{\alpha(s)}{\omega^1(s)}\right)-\left[\frac{\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]}{1+x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]}\right]}\end{aligned}$$

or

$$\bar{\omega}_1^0\left(\bar{U}_i\right)=\frac{\left[\frac{2x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]}{1+x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]}\right]\left[\omega^0\left(\sum_{s=1}^2\frac{\alpha(s)}{\omega^1(s)}\right)-2\left(\omega^0-\widehat{x}_0^0\left(\bar{U}_i\right)\right)\right]}{\left(\sum_{s=1}^2\frac{\alpha(s)}{\omega^1(s)}\right)-\left[\frac{\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]}{1+x_1^1\left[\alpha\left(\frac{1}{\omega^1(1)-x_1^1}\right)^{\sigma}+(1-\alpha)\left(\frac{1}{\omega^1(2)-x_1^1}\right)^{\sigma}\right]}\right]}\tag{25}$$

which constitute the desired expressions. ■

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