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Targeting in Adaptive Networks

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Abstract

This paper studies optimal targeting policies, consisting of eliminating (preserving) a set of agents in a network and aimed at minimizing (maximizing) aggregate effort levels. Different from the existing literature, we allow the equilibrium network to adapt after a network intervention and consider targeting of multiple agents. A simple and tractable adjustment process is introduced. We find that allowing the network to adapt may overturn optimal targeting results for a fixed network and that congestion/competition effects are crucial to understanding differences between the two settings.

Key Words: Targeting, key player policy, peer effects, local strategic complements, global strategic substitutes, positive externalities, negative externalities.
JEL Codes: D62, D85.

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1 Introduction

Important policy considerations are related to optimal targeting and, more precisely, the elimination and preservation of agents in a network. For the former, consider crime control. Law enforcement agencies may wish to minimize aggregate crime levels by removing agents from a crime network. For the latter, consider production choices of firms engaging in bilateral R&D agreements. In times of crisis, governments may wish to keep aggregate production levels as high as possible by saving a subset of firms in financial distress. We show in this paper that optimal targeting in an adaptive network may be very different from optimal targeting in a fixed network. Not taking changes in agents' linking behavior into account may imply not only suboptimal policies, but can have unintended and undesired consequences. In crime networks, for example, following an optimal targeting policy under the assumption that the network is fixed, when in fact it adapts, may lead to an increase in aggregate crime levels. We show that at the heart of differences between optimal targeting in fixed vs. adaptive networks lie congestion and competition effects. A more detailed discussion of targeting policies, including their implementability in practice, is provided in a separate section toward the end of the paper.

To study targeting policies in an adaptive network, we assume the linear quadratic payoff specification proposed in Ballester et al. (2006), which is also commonly used to study crime and R&D networks. Agents are embedded in a network and can choose a continuous, non-negative effort level. Locally, effort levels are strategic complements and induce positive externalities, while globally effort levels are strategic substitutes and induce a negative externality.[1] A planner may eliminate up to, but not necessarily equal to, a given number of agents from a network.[2] Thereafter, a simple and tractable adaption process with the following properties ensues. Agents are assumed to be myopic and, as in pairwise Nash equilibrium and pairwise stability, link formation is separated from link deletion. In the link formation stage, any link is added that is profitable for a pair of agents in isolation, given the current network and the corresponding vector of Nash equilibrium effort levels. In the link deletion stage, agents best respond to the current network by deleting any subset of links. In between link formation and link deletion stages, agents adjust their effort levels to the Nash equilibrium effort level of the current network, i.e., after links were added or deleted. We call this process pairwise best response dynamics. An optimal targeting policy then prescribes eliminating sets of agents such that the sum of discounted effort levels is minimal, where we allow for the option to not intervene at all.

The initial configuration is assumed to be a pairwise Nash equilibrium. These equilibria are relevant for two reasons. First, pairwise best response dynamics only converge to a pairwise Nash

[1] We provide formal derivations of the payoff function for the two main applications in the appendix.

[2] Optimal elimination policies with multiple agents are known to be a particularly difficult problem for fixed networks. Ballester et al. (2010) show that finding a "key group" is NP-hard. The authors provide a greedy algorithm to find such a group.

equilibrium. Second, all pairwise Nash equilibria display nested split graphs (see Hiller, 2020).[3] Nested split graphs are a particular type of core-periphery network in which agents with a higher number of links are connected to all agents to which agents with fewer links are connected. These networks are of interest in our context because they have empirical support for the applications considered.[4] Note that we assume that agents do not anticipate being targeted. The planner's targeting decision therefore does not enter agents' payoff functions. Although interesting, we follow the literature in this respect and the latter considerations are outside the scope of the present paper.[5]

We show that if the parameter governing global substitution effects is sufficiently small (and the network is not empty), then optimal targeting boils down to the following simple policy: Eliminate the maximum number of agents and target those agents that are most central. In a nested split graph the most central agents are also the ones with the highest number of links and display the highest effort levels. That is, in this case the optimal policy provides a sufficient statistic that does not require knowledge of the whole network. The intuition for this result is that with small substitution/congestion effects no pair of agents finds it profitable to create a link in any time period after the elimination, while agents' incentives to delete links are largest. The network is (weakly) sparsest and consists of the minimal number of agents in any time period. One can show that discounted aggregate effort levels are therefore also lowest. The optimal targeting policy then coincides with the case when the network does not adapt. Note, however, that this is due to the absence of targeting cost. Not taking subsequent link deletions into account underestimates the effect of targeting on aggregate effort levels. Therefore, with costly targeting, the optimal policy may differ from the fixed network case even in the absence of substitution effects. We provide an example in the main part of the paper (Example 1). If, on the other hand, the parameter governing global substitution effects is large, then agents may have incentives to create new links in subsequent periods. In particular, it may now not be optimal to target the most central agents. We first provide a sufficient condition such that, if the parameter governing global substitution effects is sufficiently large, then there exists a pairwise Nash equilibrium for which it is optimal to eliminate an agent that is the least central (displays the lowest number of links). We then focus on the star network and provide a characterization for the case when the parameter governing global substitution effects is equal to the parameter governing local complementarities. For simplicity we assume that up to two agents may be eliminated and that the discount factor is close to 1. We show that if only one agent can be eliminated, then it is always optimal to eliminate the central agent. But when two agents can be eliminated, it may

[3] Hiller (2020) also shows that a pairwise Nash equilibrium always exists for the parameter values considered.

[4] For crime networks, see Canter (2004), Dorn and South (1990), Dorn, Murji and South (1992), Ruggiero and South (1997) and Johnston (2000). For R&D networks see Tomasello et al. (2017), Kitsak et al. (2010), Rosenkopf and Schilling (2007).

[5] See for example Ballester et al. (2006), Galeotti et al. (2020), Demange (2017).

be optimal to eliminate only one agent, rather than two. The intuition is that eliminating two agents may lead to more link additions. A denser network of fewer agents may then yield higher aggregate effort levels than a sparser network with more agents. For the same reason, applying the optimal targeting policy for a fixed network, when in fact the network adapts, may lead to worse outcomes than not intervening at all. Finally, we consider the case in which a planner aims to maximize the discounted sum of aggregate effort levels and can preserve a subset of agents in distress. We show that, if global substitution effects are sufficiently small, then, in analogy to our previous results, it is optimal to save the distressed agents that are most central.

Next we briefly discuss our adaption process more generally. Note first that there exists a strand of the literature that takes a dynamic and long-run approach to network formation (for two-sided network formation, see Jackson and Watts, 2002a and 2002b).[6] These papers assume an adjustment process with noise and characterize the limit invariant distribution. Note, however, that the distribution is independent of initial conditions. That is, it is not possible to address questions related to the targeting of different agents in a network. In contrast, the simple adjustment process presented here allows us to provide analytical results and to easily calculate numerical examples. Regarding convergence properties, recall that pairwise best response dynamics only converge to pairwise Nash equilibria. If global strategic substitutes are sufficiently small, then it follows directly that pairwise best response dynamics converge to a pairwise Nash equilibrium. The reason is that no new links are created in any time period, while agents may delete links. The process is therefore bounded below by the empty network, which guarantees that the process converges. For the case when global substitution effects are large, we show convergence when needed and do not provide general convergence results. However, a one-sided link formation version of the model, for which the corresponding adaption process always converges, is presented in the online appendix.[7] There, we also provide an example that shows that intervening may be detrimental with large congestion effects. This is in the spirit of Proposition 4 of the main part of the paper. The reason to not adopt the one-sided approach is that the applications are generally thought to be two-sided. Moreover, introducing directed links introduces additional complications, without adding any interesting insights for the applications considered.

Our paper contributes to the literature on targeting, which is an active and growing area of study, not only in economics, but also in computer science and sociology.[8] Ballester et al. (2006) is an early contribution in economics and the authors characterize the optimal elimination policy of a single agent on a fixed network.[9] Galeotti et al. (2020) present a model in which a

[6]For one-sided network formation, see Bala and Goyal (2000).

[7]More precisely, one can show that the game is a potential game and we can therefore rely on the convergence results under fictitious play, as shown in Shapley and Monderer (1996).

[8]For introductions, see Borgatti (2006) and Valente (2012).

[9]For a good review of the literature on key players, see Zenou (2016).

budget constrained planner can alter the incentives of agents to exert effort at a cost on a fixed network. Effort levels are either strategic complements or strategic substitutes. The authors then characterize the set of agents to be targeted via a decomposition into principal components, which are determined by the network.[10] Belhaj and Deroian (2019) solve a principal agent problem in which a planner contracts with agents on a fixed network. It is shown that the principal may not contract with all agents and that the planner may refrain from contracting with agents that are most central. Our analysis can be thought of as complementary to this body of work and, to the best of our knowledge, the present paper is the first to provide analytic results for targeting policies when the network may adapt and to highlight the role of competition/congestion effects.[11]

The paper is organized as follows. Section 2 provides the model description. Section 3 presents our main results. Section 4 provides a more detailed discussion of our main applications, including the implementability of targeting policies. Section 5 concludes. A formal derivation of the payoff function for crime and R&D networks is provided in the appendix. In the online appendix, we relate our payoff function in detail to Ballester et al. (2006), present the one-sided link formation model and provide the proof of Proposition 3. All remaining proofs are relegated to the appendix.

2 The Model

We assume that the initial configuration is a pairwise Nash equilibrium and start by defining the corresponding network formation game. Note that the deviations considered are also the relevant deviations for our pairwise best response dynamics. Moreover, effort levels at each time period correspond to the Nash equilibrium effort levels given the prevailing network. We then introduce pairwise best response dynamics and further useful concepts formally.

2.1 Payoffs, Pairwise Nash Equilibrium and Some Notation

Let $N = \{1, 2, ..., n\}$ be the set of players with $n \geq 3$. Each agent i chooses an effort level $x_i \in X$ and announces a set of agents to whom the agent wishes to be linked, which is represented by a row vector $\mathbf{g}_i = (g_{i,1}, ..., g_{i,i-1}, g_{i,i+1}, ..., g_{i,n})$, with $g_{i,j} \in \{0, 1\}$ for each $j \in N \backslash \{i\}$. An entry $g_{i,j} = 1$ in $\mathbf{g}_i$ is interpreted as agent i announcing a link to agent j, while an entry $g_{i,j} = 0$ in $\mathbf{g}_i$ is taken to mean that agent i does not announce a link to agent j. Assume $X = [0, +\infty)$ and $\mathbf{g}_i \in G_i = \{0, 1\}^{n-1}$. The set of agent i's strategies is denoted by $S_i = X \times G_i$ and the set of strategies of all players by $S = S_1 \times S_2 \times ... \times S_n$. A strategy profile $\mathbf{s} = (\mathbf{x}, \mathbf{g}) \in S$ then specifies each player's individual effort level, $\mathbf{x} = (x_1, x_2, ..., x_n)$, and intended links, $\mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_n)$.

[10] For a model with targeting on a fixed network and strategic complementarities, also see Demange (2017).

[11] Targeting is also studied in models of optimal pricing with interdependent consumers. See, for example, Bloch and Querou (2013) and Fainmesser and Galeotti (2017). A further strand of the literature is concerned with seeding and diffusion in networks. See Galeotti and Goyal (2009) and Galeotti and Rogers (2013).

A link between agents i and j, denoted by $\bar{g}_{i,j} = 1$, is created if and only if *both* agents i and j announce the link. That is, $\bar{g}_{i,j} = 1$ if and only if $g_{i,j} = g_{j,i} = 1$ (and $\bar{g}_{i,j} = 0$ otherwise) and therefore $\bar{g}_{i,j} = \bar{g}_{j,i}$. The undirected graph $\bar{\mathbf{g}}$ is defined as $\bar{\mathbf{g}} = \{\{i,j\} \in N : \bar{g}_{i,j} = 1\}$. That is, $\bar{\mathbf{g}}$ is a collection of links, which are listed as subsets of N of size 2. Denote the set of i's neighbors in $\bar{\mathbf{g}}$ with $N_i(\bar{\mathbf{g}}) = \{j \in N : \bar{g}_{i,j} = 1\}$ and the corresponding cardinality with $\eta_i(\bar{\mathbf{g}}) = |N_i(\bar{\mathbf{g}})|$.[12] Given a network $\bar{\mathbf{g}}$, $\bar{\mathbf{g}} + \bar{g}_{i,j}$ and $\bar{\mathbf{g}} - \bar{g}_{i,j}$ have the following interpretation. When $\bar{g}_{i,j} = 0$ in $\bar{\mathbf{g}}$, then $\bar{\mathbf{g}} + \bar{g}_{i,j}$ adds the link $\bar{g}_{i,j} = 1$, while if $\bar{g}_{i,j} = 1$ in $\bar{\mathbf{g}}$, then $\bar{\mathbf{g}} + \bar{g}_{i,j} = \bar{\mathbf{g}}$. Similarly, if $\bar{g}_{i,j} = 1$ in $\bar{\mathbf{g}}$, then $\bar{\mathbf{g}} - \bar{g}_{i,j}$ deletes the link $\bar{g}_{i,j}$, while if $\bar{g}_{i,j} = 0$ in $\bar{\mathbf{g}}$, then $\bar{\mathbf{g}} - \bar{g}_{i,j} = \bar{\mathbf{g}}$.

In order to compare different networks we write $\bar{\mathbf{g}} \subset \hat{\bar{\mathbf{g}}}$ to indicate that $\{\{i,j\} \in N : \{i,j\} \in \bar{\mathbf{g}}\} \subset \{\{i,j\} \in N : \{i,j\} \in \hat{\bar{\mathbf{g}}}\}$. Similarly, we write $\bar{\mathbf{g}} = \hat{\bar{\mathbf{g}}}$ for $\{\{i,j\} \in N : \{i,j\} \in \bar{\mathbf{g}}\} = \{\{i,j\} \in N : \{i,j\} \in \hat{\bar{\mathbf{g}}}\}$, and $\bar{\mathbf{g}} \subseteq \hat{\bar{\mathbf{g}}}$ for $\{\{i,j\} \in N : \{i,j\} \in \bar{\mathbf{g}}\} \subseteq \{\{i,j\} \in N : \{i,j\} \in \hat{\bar{\mathbf{g}}}\}$. The network is called empty and denoted by $\bar{\mathbf{g}}^e$ if $\bar{g}_{i,j} = 0$ $\forall i,j \in N$, while it is called complete and denoted by $\bar{\mathbf{g}}^c$ if $\bar{g}_{i,j} = 1$ $\forall i,j \in N$ such that $i \neq j$.

Payoffs to player i under strategy profile $\mathbf{s} = (\mathbf{x}, \mathbf{g})$ are given by

$$\Pi_i(\mathbf{s}) = \pi_i(\mathbf{x}, \bar{\mathbf{g}}) - \eta_i(\bar{\mathbf{g}})\kappa,$$

where κ denotes linking cost with $\kappa > 0$. Gross payoffs, i.e. payoffs excluding linking cost, $\pi_i(\mathbf{x}, \bar{\mathbf{g}})$, are given by the frequently employed linear-quadratic payoff function with local complementarities and global substitutes (Ballester et al., 2006). That is,

$$\pi_i(\mathbf{x}, \bar{\mathbf{g}}) = \alpha x_i - \tfrac{1}{2}(\beta + \gamma)x_i^2 + \lambda x_i \textstyle\sum_{j \in N_i(\bar{\mathbf{g}})} x_j - \gamma x_i \sum_{j \in N \setminus \{i\}} x_j \;\; \forall i \in N.$$

To guarantee existence and uniqueness of a Nash equilibrium in effort levels for any fixed network $\bar{\mathbf{g}}$, we can resort to Ballester et al. (2006) and assume that $\beta > (n-1)\lambda$. The best response and value function are defined as follows.

Best response function. The unique best response of player i to the vector of effort levels $\mathbf{x}_{-i}$ in network $\bar{\mathbf{g}}$ is given by

$$\bar{x}_i(\mathbf{x}_{-i}, \bar{\mathbf{g}}) = \bar{x}_i(y_i(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})) = \tfrac{1}{\beta+\gamma}\left(\alpha + \lambda \textstyle\sum_{j \in N_i(\bar{\mathbf{g}})} x_j - \gamma \sum_{j \in N \setminus \{i\}} x_j\right).$$

Value function. The maximized gross payoff for $\mathbf{x}_{-i}$ in network $\bar{\mathbf{g}}$ is given by

$$\pi_i(\bar{x}_i, \mathbf{x}_{-i}, \bar{\mathbf{g}}) = v_i(y_i(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})) = \tfrac{1}{2(\beta+\gamma)}\left(\alpha + \lambda \textstyle\sum_{j \in N_i(\bar{\mathbf{g}})} x_j - \gamma \sum_{j \in N \setminus \{i\}} x_j\right)^2.$$

The aggregate effort level of agent i's neighbors in $\bar{\mathbf{g}}$, which we sometimes call agent i's effort level "accessed", is written as $y_i(\bar{\mathbf{g}}) = \sum_{j \in N_i(\bar{\mathbf{g}})} x_j$. The aggregate effort level of all agents other

[12] Note that agents are not linked to themselves and therefore not included in their own neighborhood.

than i is written as $z_i(\bar{\mathbf{g}}) = \sum_{j \in N \setminus \{i\}} x_j$. We sometimes write x_i, y_i, z_i, π_i, $\bar{x}_i$ and v_i to simplify notation. Likewise, at times we drop the subscripts when it is clear from the context.

Next we define *pairwise Nash equilibrium* (PNE). When agents i and j deviate to create a link, then deviation effort levels are assumed to be mutual best responses (while the remaining agent's effort levels remain unchanged). The corresponding deviation effort levels are denoted by $x_i' = \bar{x}(y_i(\bar{\mathbf{g}}) + x_j', z_i(\bar{\mathbf{g}}) + x_j' - x_j)$. We sometimes write $x_i'(\bar{\mathbf{g}} + \bar{g}_{i,j})$ to denote agent i's deviation effort level when creating a link with agent j.

A strategy profile $\mathbf{s} = (\mathbf{x}, \mathbf{g})$ is a pairwise Nash equilibrium *iff*

- for any $i \in N$ and every $\mathbf{s}_i \in S_i$, $\Pi_i(\mathbf{s}) \geq \Pi_i(\mathbf{s}_i, \mathbf{s}_{-i})$;
- for all $\bar{g}_{i,j} = 0$, if $\Pi_i(x_i', x_j', \mathbf{x}_{-i,-j}, \bar{\mathbf{g}} + \bar{g}_{i,j}) > \Pi_i(\mathbf{s})$,

  then $\Pi_j(x_i', x_j', \mathbf{x}_{-i,-j}, \bar{\mathbf{g}} + \bar{g}_{i,j}) < \Pi_j(\mathbf{s})$.

A pairwise Nash equilibrium is both a Nash equilibrium and pairwise stable and therefore refines Nash equilibrium. Pairwise Nash equilibrium allows for deviations where a pair of agents creates a link (and deviating agents best respond to each other's effort level). Furthermore, pairwise Nash equilibrium allows for deviations in which an agent deletes any subset of existing links (and adjusts her effort level). However, deviations where a pair of agents creates a link and/or adjusts effort levels and *simultaneously* deletes any subset of existing links are not considered. We write $(\mathbf{x}(\bar{\mathbf{g}}), \bar{\mathbf{g}})$ to denote a network $\bar{\mathbf{g}}$ and the corresponding vector of Nash equilibrium effort levels, $\mathbf{x}(\bar{\mathbf{g}})$. The configuration $(\mathbf{x}(\bar{\mathbf{g}}), \bar{\mathbf{g}})$ is a pairwise Nash equilibrium if and only if the above conditions are satisfied for all agents/pairs of agents.

Hiller (2020) shows that if $\lambda \geq \gamma \geq 0$ and $\lambda > 0$, then any pairwise Nash equilibrium network is a nested split graph and a pairwise Nash equilibrium always exists. These assumptions are in line with the applications considered and we adopt them throughout the paper.[13] A formal definition of a nested split graph and the corresponding equilibrium characterization are presented below.

Definition 1: A network $\bar{\mathbf{g}}$ is a *nested split graph* if and only if

$$[\bar{g}_{i,l} = 1 \textit{ and } \eta_k(\bar{\mathbf{g}}) \geq \eta_l(\bar{\mathbf{g}})] \Rightarrow \bar{g}_{i,k} = 1.$$

Proposition 1 (Hiller, 2020). *In any PNE,* $(\mathbf{x}, \bar{\mathbf{g}})$*, the network* $\bar{\mathbf{g}}$ *is a nested split graph such that* $x_i < x_k \Leftrightarrow \eta_i(\bar{\mathbf{g}}) < \eta_k(\bar{\mathbf{g}}) \Leftrightarrow v(y_i(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})) < v(y_k(\bar{\mathbf{g}}), z_k(\bar{\mathbf{g}}))$ *holds.*

It is sometimes convenient to write equilibrium and deviation payoffs as a function of γ and we adopt the following notation. We write $\mathbf{x}(\bar{\mathbf{g}}, \gamma)$ for the vector of Nash equilibrium effort levels

[13] For example, for the case of R&D networks, König et al. (2019) provide estimates for λ and γ and show that $\lambda > \gamma > 0$. See Pattacchini and Zenou (2012) for peer effects in crime.

as a function of the network and γ, while suppressing the remaining parameters. Likewise, we denote agent i's Nash equilibrium effort level by $x_i(\bar{\mathbf{g}}, \gamma)$ and write agent i's deviation effort level when creating a new link with agent j as $x'_i(\bar{\mathbf{g}}+\bar{g}_{i,j}, \gamma)$. With some abuse of notation we write equilibrium gross payoffs as $v_i(\bar{\mathbf{g}}, \gamma)$. We denote the (gross) marginal deviation payoffs of agent i when creating a link with agent j with $\Delta v_i(\bar{\mathbf{g}} + \bar{g}_{i,j}, \gamma)$, where $\Delta v_i(\bar{\mathbf{g}} + \bar{g}_{i,j}, \gamma) = v_i(\bar{\mathbf{g}} + \bar{g}_{i,j}, \gamma) - v_i(\bar{\mathbf{g}}, \gamma)$.

For our analysis it is useful to define paths and components. A path of length k from i to j is a sequence $i_0, ..., i_k$ of players such that $i_0 = i$ and $i_k = j$, $i_p \neq i_{p+1}$ and $\bar{g}_{i_p,i_{p+1}} = 1$ for all $0 \leq p \leq k-1$. Components are maximal subsets of agents $N^s \subseteq N$, such that for every $i, j \in N^s$ (with $i \neq j$), there exists a path between i and j.[14] A component N^s is called complete if a link is present for every $i, j \in N^s$ (with $i \neq j$). Note that the binary relationship of "being connected by a network path" is an equivalence relationship and therefore components partition the set of agents. We write $k \in N^i(\bar{\mathbf{g}})$ to denote that agent k lies in agent i's component in network $\bar{\mathbf{g}}$ (and there therefore exists a path between agents i and k in $\bar{\mathbf{g}}$). A network is said to be connected if there exists only one component.[15]

2.2 Optimal Targeting Policies: Preliminaries

In the following we present a simple infinite horizon adjustment process, which we call pairwise best response dynamics. Pairwise best response dynamics mirror pairwise stability and pairwise Nash equilibrium in the following sense: pairs of agents can either create links, or delete links, but the process does not allow for the simultaneous creation and deletion of links. We then define optimal targeting policies formally and introduce a relabeling procedure, which allows us to compare networks after different sets of agents were eliminated from the network. Finally, we introduce minimal deletion best responses, which simplify the exposition but are not necessary for our results.

Starting from a pairwise Nash equilibrium, $(\mathbf{x}(\bar{\mathbf{g}}), \bar{\mathbf{g}})$, denote the network after a set of agents E is eliminated from $\bar{\mathbf{g}}$ by $\bar{\mathbf{g}}_0^{-E}$. Agents adjust their efforts to the Nash equilibrium effort levels of the new network $\bar{\mathbf{g}}_0^{-E}$, which yields $\mathbf{x}(\bar{\mathbf{g}}_0^{-E})$. In time period one, given the configuration $(\mathbf{x}(\bar{\mathbf{g}}_0^{-E}), \bar{\mathbf{g}}_0^{-E})$, all links are added to the network that constitute a profitable deviation in isolation.[16] That is, any link not already present, such that a pair of agents can profitably deviate by creating a link, given $\mathbf{x}(\bar{\mathbf{g}}_0^{-E})$ and $\bar{\mathbf{g}}_0^{-E}$, is added to the network $\bar{\mathbf{g}}_0^{-E}$. This yields the network $\bar{\mathbf{g}}_1^{-E}$. Agents again update their effort levels to Nash equilibrium effort levels, $\mathbf{x}(\bar{\mathbf{g}}_1^{-E})$, given the network $\bar{\mathbf{g}}_1^{-E}$. In time period two agents best respond by deleting links given the configuration

[14] Singleton agents are assumed to form trivial components of size one.

[15] For definitions of paths and components also see Jackson (2010) and Vega-Redondo (2007).

[16] From the definition of pairwise Nash equilibrium, for a link between i and j to be profitable, i and j's deviation payoff of creating a link among themselves in $(\mathbf{x}(\bar{\mathbf{g}}_0^{-E}), \bar{\mathbf{g}}_0^{-E})$ is weakly positive for both agents and strictly positive for at least one agent.

$(\mathbf{x}(\bar{\mathbf{g}}_1^{-E}), \bar{\mathbf{g}}_1^{-E})$, which yields the network $\bar{\mathbf{g}}_2^{-E}$ and corresponding Nash equilibrium effort levels $\mathbf{x}(\bar{\mathbf{g}}_2^{-E})$.[17] The link creation and link deletion stages of time period one and two are then assumed to alternate indefinitely. Below we provide a summary of our adjustment process.

Start with $\bar{\mathbf{g}}_0^{-E}$ and $\mathbf{x}(\bar{\mathbf{g}}_0^{-E})$ and define $\bar{\mathbf{g}}_t^{-E}$ and $\mathbf{x}(\bar{\mathbf{g}}_t^{-E})$ iteratively as follows.

Pairwise best response dynamics:

Step 1: Add all profitable links given $\bar{\mathbf{g}}_t^{-E}$ and $\mathbf{x}(\bar{\mathbf{g}}_t^{-E}) \Rightarrow \bar{\mathbf{g}}_{t+1}^{-E}$.

Step 2: Allow for link deletions given $\bar{\mathbf{g}}_{t+1}^{-E}$ and $\mathbf{x}(\bar{\mathbf{g}}_{t+1}^{-E}) \Rightarrow \bar{\mathbf{g}}_{t+2}^{-E}$

Repeat Steps 1 and 2 ad infinitum, letting $t \to \infty$.

Finally, an optimal targeting policy identifies the set of agents for which the discounted sum of aggregate effort levels is minimal.[18] Denote the discount factor by δ and denote by $\mathscr{P}(N)$ the power set of N. Denote the set of eliminated agents by E, with $E \in \mathscr{P}(N)$. The maximum number of agents that can be eliminated from the network is given by e with $1 \leq e < n$. Note that this allows for the possibility that no agent is targeted.

Optimal targeting policy: Pick a set of agents E such that

$$min_{E \subset N}\{\sum_{t=0}^{\infty} \delta^t \sum_{j \in N \setminus E} x_j(\bar{\mathbf{g}}_t^{-E}) \mid \ E \in \mathscr{P}(N) \ and \ \mid E \mid \leq e\}.$$

Relabeling Procedure

In order to be able to compare different networks after different sets of agents (with same cardinality) are eliminated from the network, we need to relabel agents so that the sets of the remaining agents are the same. To illustrate this, consider first the case when only one agent is eliminated and assume we want to compare $\bar{\mathbf{g}}^{-k}$ and $\bar{\mathbf{g}}^{-l}$. We can then relabel agent l in $\bar{\mathbf{g}}^{-k}$ as, say, r, while we relabel agent k in $\bar{\mathbf{g}}^{-l}$ as r. This yields the same set of agents in $\bar{\mathbf{g}}^{-k}$ and $\bar{\mathbf{g}}^{-l}$, i.e. $\{N \setminus \{k, l\}\} \cup \{r\}$, which then allows us to directly compare the two networks. The relabeling procedure is more complicated when considering the elimination of multiple agents. Assume that different sets of agents, E_i and E_j, are eliminated from $\bar{\mathbf{g}}$ and denote the resulting networks with $\bar{\mathbf{g}}^{-E_i}$ and $\bar{\mathbf{g}}^{-E_j}$. We will be interested in the deletion of sets of equal size and therefore assume that $\mid E_i \mid = \mid E_j \mid$. Note that an agent $k \in N$ such that $k \in E_i \cap E_j$ is eliminated in both $\bar{\mathbf{g}}^{-E_i}$ and $\bar{\mathbf{g}}^{-E_j}$ and we can therefore disregard any such agent in our relabeling procedure. Define $\tilde{E}_i$ and $\tilde{E}_j$ as follows, $\tilde{E}_i = E_i \setminus E_j$ and, similarly, $\tilde{E}_j = E_j \setminus E_i$. Denote $\tilde{e} = \mid \tilde{E}_i \mid = \mid \tilde{E}_j \mid$. Next, choose any ranking of agents in $\tilde{E}_i$ such that $\eta_k^{(1,\tilde{E}_i)}(\bar{\mathbf{g}}) \leq \eta_l^{(2,\tilde{E}_i)}(\bar{\mathbf{g}}) \leq ... \leq \eta_m^{(\tilde{e},\tilde{E}_i)}(\bar{\mathbf{g}})$, where the subscript

[17] Note that we assume that agents best respond via minimal deletion best responses, which are defined formally only later. While this assumption simplifies the analysis and notation, it should be noted that our results do not depend on it.

[18] Alternatively one could define the optimal targeting policy in terms of the overtaking criterion. Note that our results go through for the overtaking criterion as well.

denotes the label of the agent in $\bar{\mathbf{g}}$, the first entry in the superscript denotes the agent's position in the chosen ranking and the second entry signifies that the agent is in the set $\tilde{E}_i$. Similarly, choose any ranking of agents in $\tilde{E}_j$ such that $\eta_s^{(1,\tilde{E}_j)}(\bar{\mathbf{g}}) \leq \eta_t^{(2,\tilde{E}_j)}(\bar{\mathbf{g}}) \leq ... \leq \eta_u^{(\tilde{e},\tilde{E}_j)}(\bar{\mathbf{g}})$. To simplify notation, we sometimes drop the subscript. Note that $\tilde{E}_i \cap \tilde{E}_j = \emptyset$ by construction. Agents are then relabeled in the following way. For a pair of agents k and s with the same superscript x, $\eta_k^{(x,\tilde{E}_i)}(\bar{\mathbf{g}})$ and $\eta_s^{(x,\tilde{E}_j)}(\bar{\mathbf{g}})$, we relabel s as r_x in $\bar{\mathbf{g}}^{-E_i}$ and k as r_x in $\bar{\mathbf{g}}^{-E_j}$. We write $r_x(\bar{\mathbf{g}}^{-E_i})$ to denote agent r_x in network $\bar{\mathbf{g}}^{-E_i}$ and write $r_x(\bar{\mathbf{g}}^{-E_j})$ for agent r_x in network $\bar{\mathbf{g}}^{-E_j}$. Note also that the agent relabeled in each instance is the agent that is *not* eliminated in the respective network: s in the case of $\bar{\mathbf{g}}^{-E_i}$ and k in the case of $\bar{\mathbf{g}}^{-E_j}$. Define the set of agents after elimination and relabeling as $N(E_i)$ and $N(E_j)$ and note that $N(E_i) = N(E_j) = \{N \setminus \{E_i \cup E_j\}\} \cup \{r_1, r_2, ..., r_{\tilde{e}}\}$. This allows us to compare the networks $\bar{\mathbf{g}}^{-E_i}$ and $\bar{\mathbf{g}}^{-E_j}$ directly.

To show our results, it is useful to define the following preorder, denoted by $\succsim$.

Definition 2: $\tilde{E}_i \succsim \tilde{E}_j$ if and only if $\eta^{(x,\tilde{E}_i)}(\bar{\mathbf{g}}) \geq \eta^{(x,\tilde{E}_j)}(\bar{\mathbf{g}})$ for all $x \in \{1, ..., \tilde{e}\}$.

Note that $\succsim$ does not depend on which particular ranking is chosen for $\tilde{E}_i$ and $\tilde{E}_j$. Define $\tilde{E}_i \sim \tilde{E}_j$ as follows: $\tilde{E}_i \sim \tilde{E}_j$ if and only if $\tilde{E}_i \succsim \tilde{E}_j$ and $\tilde{E}_j \succsim \tilde{E}_i$. Likewise, define $\tilde{E}_i \succ \tilde{E}_j$ as $\tilde{E}_i \succ \tilde{E}_j$ if and only if $\tilde{E}_i \succsim \tilde{E}_j$ and not $\tilde{E}_j \succsim \tilde{E}_i$. Finally, note that when $E_i \in \mathscr{E}(e)$ and $E_j \notin \mathscr{E}(e)$, then $\tilde{E}_i \succ \tilde{E}_j$, while if $E_i, E_j \in \mathscr{E}(e)$, then $\tilde{E}_i \sim \tilde{E}_j$.

Minimal Deletion Best Responses

To simplify the exposition we assume that agents play minimal deletion best responses. More precisely, if an agent's current linking strategy $\mathbf{g}_i$ in configuration $(\mathbf{x}(\bar{\mathbf{g}}), \bar{\mathbf{g}})$ is (part of) a best response, then we assume that the agent does not deviate and the minimal deletion best response is simply $\mathbf{g}_i$ (together with the corresponding best response effort level $\bar{x}_i(\bar{\mathbf{g}})$). If, however, the current linking strategy $\mathbf{g}_i$ is *not* part of a best response and multiple profitable deletion deviation best responses exist, then the strategy chosen is such that, loosely speaking, the number of links remaining after an agent's deviation strategy is minimal. Assuming that agents play minimal best responses allows for a cleaner analysis and to calculate examples more easily. However, it is important to note that our results do not depend on this assumption.[19]

Consider a deletion best response by agent i, written as $\mathbf{g}'_i$ and $\bar{x}_i(\mathbf{g}'_i, \bar{\mathbf{g}})$, where $\bar{x}_i(\mathbf{g}'_i, \bar{\mathbf{g}})$ is the best response effort level of agent i when deviating to linking strategy $\mathbf{g}'_i$ in $(\mathbf{x}(\bar{\mathbf{g}}), \bar{\mathbf{g}})$. To simplify notation, we often simply write $\mathbf{g}'_i$ for a deletion best response and it is implicit that agent i plays the corresponding best response effort level $\bar{x}_i(\mathbf{g}'_i, \bar{\mathbf{g}})$. Denote the network after proposed deviation by $\bar{\mathbf{g}}'_i$. Again we drop the subscript when it is clear from the context. Deviation payoffs

[19]They also go through, for example, if we were to assume that agents choose randomly among their best responses when deleting links and that the planner aims to minimize the expected discounted stream of aggregate effort levels.

are defined by $\Pi_i(\mathbf{g}'_i, \bar{\mathbf{g}}) = \pi_i(\bar{x}_i(\mathbf{g}'_i, \bar{\mathbf{g}}), \mathbf{x}_{-i}(\bar{\mathbf{g}}), \bar{\mathbf{g}}') - \eta_i(\bar{\mathbf{g}}')\kappa$. Write $\mathbf{g}'_i \subseteq \mathbf{g}_i$ if for all $j \in N \setminus \{i\}$ with $g'_{i,j} = 1$ in $\mathbf{g}'_i$, $g_{i,j} = 1$ also holds in $\mathbf{g}_i$. For configuration $(\mathbf{x}(\bar{\mathbf{g}}), \bar{\mathbf{g}})$ we then define agent i's minimal deletion best response, $\mathbf{g}'^m_i$, as follows.

Definition 3: Minimal Deletion Best Responses.

If $\mathbf{g}_i \in argmax_{\mathbf{g}'_i:\, \mathbf{g}'_i \subseteq \mathbf{g}_i} \Pi_i(\mathbf{g}'_i, \bar{\mathbf{g}})$, then $\mathbf{g}'^m_i : \mathbf{g}'^m_i = \mathbf{g}_i$.

If $\mathbf{g}_i \notin argmax_{\mathbf{g}'_i:\, \mathbf{g}'_i \subseteq \mathbf{g}_i} \Pi_i(\mathbf{g}'_i, \bar{\mathbf{g}})$, then $\mathbf{g}'^m_i$: i) $\mathbf{g}'^m_i \in argmax_{\mathbf{g}'_i:\, \mathbf{g}'_i \subseteq \mathbf{g}_i} \Pi_i(\mathbf{g}'_i, \bar{\mathbf{g}})$ and

ii) $\mathbf{g}'^m_i \subseteq \mathbf{g}'_i \ \forall \mathbf{g}'_i \in argmax_{\mathbf{g}'_i:\, \mathbf{g}'_i \subseteq \mathbf{g}_i} \Pi_i(\mathbf{g}'_i, \bar{\mathbf{g}})$.

That is, if $\mathbf{g}_i$ is (part of) a best response in configuration $(\mathbf{x}(\bar{\mathbf{g}}), \bar{\mathbf{g}})$, then $\mathbf{g}'^m_i = \mathbf{g}_i$, while if $\mathbf{g}_i$ is not (part of) a best response, then $\mathbf{g}'^m_i$ selects the deletion best response that can be considered minimal. A minimal deletion best response always exists and is unique (Lemma 7). Note that, due to strategic complementarities, $\bar{x}_i(\mathbf{g}'_i, \bar{\mathbf{g}})$ is then also minimal. We denote the network after a minimal deletion best response by agent i in $\bar{\mathbf{g}}$ with $\bar{\mathbf{g}}'^m_i$. Again, we drop the subscript when it is clear from the context. Finally, denote the set of agents to which i deletes a link in $\bar{\mathbf{g}}$ with $D_i(\mathbf{g}'_i, \bar{\mathbf{g}}) = \{j \in N : j \in N_i(\bar{\mathbf{g}}) \; and \; j \notin N_i(\bar{\mathbf{g}}')\}$.

3 Optimal Targeting Policies

We consider the problem of a planner, who aims to minimize the discounted flow of aggregate effort levels by eliminating agents from the network. We first present results for the case when competition or congestion effects are small.

3.1 Small Competition/Congestion Effects

Proposition 2 shows that, if the parameter governing global substitution effects is sufficiently small, then optimal targeting prescribes eliminating e agents with the highest number of links. More precisely, the optimal targeting policy is characterized in terms of a family of sets $\mathscr{E}(e)$, which consists of all sets of agents E_i, such that any agent in E_i displays a (weakly) higher number of links than any agent not in the set. A formal definition is provided below.

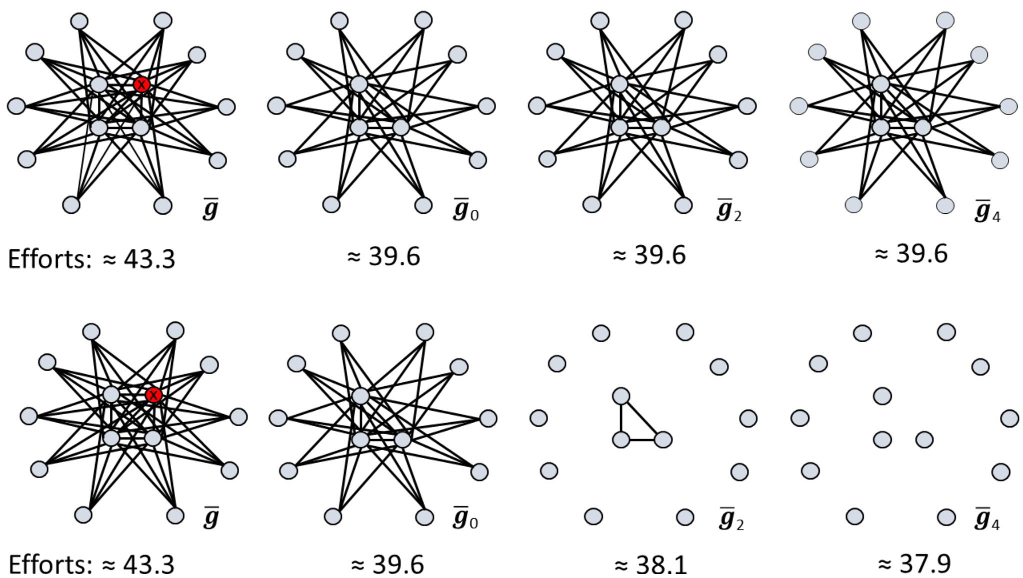
Definition 4: $\mathscr{E}(e) = \{E_i \subset N : \mid E_i \mid = e \; and \; \eta_j(\bar{\mathbf{g}}) \geq \eta_k(\bar{\mathbf{g}}) \; \forall j, k : j \in E_i, \; k \notin E_i\}$.

For Proposition 2 we assume that the maximum number of agents that can be eliminated, e, is such that it is not possible to obtain the empty network in $t = 0$ via the elimination of agents. The reason for this assumption is that when e is larger than this threshold, denoted by $t(\bar{\mathbf{g}})$ (and derived in Lemma 1), then for γ sufficiently small any set of agents E such that $\bar{\mathbf{g}}^{-E}$ is the empty network is an optimal targeting policy, including any $E_i \in \mathscr{E}(e)$. To simplify the exposition we therefore focus on the case when $e < t(\bar{\mathbf{g}})$. Moreover, we assume that the network is not empty and only briefly comment on this case here. If $\bar{\mathbf{g}}$ is empty, then one can show easily that for $\gamma = 0$ no new links will be formed in any time period and optimal targeting prescribes eliminating any set of e agents. If, however, $\gamma > 0$, then marginal benefits of creating links in the empty network are larger, the smaller the number agents. That is, for any $\gamma > 0$, there are linking cost such that agents find it profitable to create new links after an elimination and it may then be optimal to not intervene.

Proposition 2: *Assume $\bar{\mathbf{g}} \notin \{\bar{\mathbf{g}}^e\}$ and $e < t(\bar{\mathbf{g}})$. If γ is sufficiently small, then an optimal targeting policy prescribes eliminating a set of agents $E_i \in \mathscr{E}(e)$ for any value of δ.*

We first provide intuition and a brief summary of the main arguments used to show Proposition 2. The following result, presented in Theorem 2 in Ballester et al. (2006), will be useful for our analysis: if one network covers another, i.e. if $\bar{\mathbf{g}}_1 \subset \bar{\mathbf{g}}_2$ holds, then the sum of Nash equilibrium effort levels is strictly lower in $\bar{\mathbf{g}}_1$ than in $\bar{\mathbf{g}}_2$. First we present the arguments allowing us to show that, conditional on eliminating e' agents (with $e' \leq e < t(\bar{\mathbf{g}})$), optimal targeting prescribes eliminating a set of agents E_i such that $E_i \in \mathscr{E}(e')$. Only later is the optimal number of targeted agents discussed. We start by showing that for any pairwise Nash equilibrium network $\bar{\mathbf{g}}$ and for any e', if $E_i \in \mathscr{E}(e')$ and $E_j \notin \mathscr{E}(e')$, then $\bar{\mathbf{g}}_0^{-E_i} \subset \bar{\mathbf{g}}_0^{-E_j}$ holds (Lemma 3). That is, in time period zero, aggregate effort levels are strictly lower when eliminating a set of agents $E_i \in \mathscr{E}(e')$ than when eliminating a set of agents $E_j \notin \mathscr{E}(e')$. We next show that payoffs are continuous in γ (Lemma 4) and then, via simple best response dynamics, that for $\gamma = 0$ effort levels always decrease strictly for some agents and weakly for all agents when deleting links or eliminating agents from a network (Lemma 5 and Lemma 6). The latter results, together with the continuity of payoffs in γ, can then be shown to be relevant for γ sufficiently small as well. Moreover, for γ sufficiently small, no new links are created in any time period, no matter which set of agents is eliminated (Lemma 9 and Lemma 10). Regarding link deletions, we assume that agents select minimal deletion best responses, as outlined above. We show that if $\bar{\mathbf{g}}_0^{-E_i} \subset \bar{\mathbf{g}}_0^{-E_j}$, then in any time period t, if a link is deleted in $\bar{\mathbf{g}}_t^{-E_j}$, then it is also deleted in $\bar{\mathbf{g}}_t^{-E_i}$ (Lemma 8). That is, $\bar{\mathbf{g}}_t^{-E_i} \subseteq \bar{\mathbf{g}}_t^{-E_j}$ holds for all time periods $t \geq 0$ and the set inclusion is strict for $t = 0$. Therefore, the optimal targeting policy, conditional on e' agents being eliminated, prescribes eliminating a set of agents E_i such that $E_i \in \mathscr{E}(e')$. To show that it is optimal to eliminate e agents, we compare eliminating $E_i \in \mathscr{E}(e)$ with $E_k \in \mathscr{E}(e')$ and assume that $e' < e$. Note that the number of agents eliminated in E_i is strictly larger than in E_k. One can then show that in any time period t there exists an intermediary network that can be obtained from $\bar{\mathbf{g}}_t^{-E_i}$ by adding a set of singletons. Moreover, the intermediary network is covered by $\bar{\mathbf{g}}_t^{-E_k}$. Finally, for γ sufficiently small, aggregate effort levels in $\bar{\mathbf{g}}_t^{-E_i}$ are strictly smaller than in the corresponding intermediary network and, since the intermediary network is covered by $\bar{\mathbf{g}}_t^{-E_k}$, optimal targeting prescribes eliminating a set of agents E_i such that $E_i \in \mathscr{E}(e)$.

Note that, since the initial pairwise Nash equilibrium network $\overline{\mathbf{g}}$ is a nested split graph, the agents with the highest number of links are also those which display the highest Bonacich centrality and the highest inter-centrality. That is, as long as γ is sufficiently small and $e = 1$, the optimal targeting policy in Proposition 2 coincides with the key player policy in Ballester et al. (2006). The following example illustrates that if targeting is costly, then the optimal targeting policies in a fixed vs. an adaptive network may differ even in the absence of global substitution effects. The reason is that if the network adapts, then the initial elimination of agents may cause subsequent link deletions, leading to a sparser network and lower aggregate effort levels. Note that in evaluating the optimal targeting policy, we treat targeting cost c the same as crime effort in time period zero.



Example 1: Assume $n = 14$, $\alpha = 70$, $\beta = 24$, $\lambda = 0.2$, $\gamma = 0$, $\kappa = 1.86$, $c = 40$, $\delta = 0.9$. Then a complete core-periphery network with 4 agents in the core and 10 agents in the periphery, as depicted in $\overline{\mathbf{g}}$ above, is a pairwise Nash equilibrium. The discounted stream of effort levels is 432.5. If the network is fixed, then the discounted stream of effort levels is 396.3. With a targeting cost of 40 it is therefore optimal to not intervene. If, however, the network adapts, then in $t = 2$ core agents delete their links to all agents in the periphery, yielding a group dominant network. Then, in $t = 4$ core agents delete all their remaining links, yielding the empty network from then onward. The discounted stream of efforts is 382.7. With a targeting cost of 40 it is therefore optimal to eliminate a central agent.

We briefly provide general comments regarding Proposition 2. Note first that we propose a particularly simple adjustment process, so as to obtain a tractable model and to be able to calculate examples easily. However, one could in principle assume a different adjustment process, where, for example, agents/pairs of agents are chosen at random to update their strategy. Since the initial network is minimal when eliminating a set of agents $E_i \in \mathscr{E}(e)$ and it is never profitable to create links, in expectation the network and aggregate effort levels are minimal at each time period as well. That is, the optimal targeting policy characterized in Proposition 2 also applies. Moreover, our result does not depend on whether the process starts with a link creation or a link deletion stage. To see this, note that no links are created in any time period and, if $\overline{\mathbf{g}}_0^{-E_i} \subset \overline{\mathbf{g}}_0^{-E_j}$, then again in any time period (weakly) more links are deleted in $\overline{\mathbf{g}}_t^{-E_i}$ than in $\overline{\mathbf{g}}_t^{-E_j}$. Finally, since there can only be link deletions and there are no link additions, and since the number of links is bounded below by zero, the adjustment process converges. In fact, it converges to a nested split graph.[20]

Having studied the elimination of multiple agents allows us to provide a corollary for Proposition 2. Assume the planner is concerned with maximizing, rather than minimizing, the discounted stream of aggregate effort levels. Moreover, assume that, rather than eliminating a set of agents, the planner is now concerned with saving a set of firms S, with cardinality $\mid S \mid \leq s$, which in turn is a subset of failing firms, F, with $\mid F \mid = f$. That is, $S \subseteq F \subseteq N$ and $1 \leq s \leq f$. Assume further that failing firms close down if they are not saved. Define the set $\mathcal{S}(s) = \{S_i \subseteq F : \mid S_i \mid = s \textit{ and } \eta_j(\overline{\mathbf{g}}) \geq \eta_k(\overline{\mathbf{g}}) \ \forall j, k \in F : j \in S_i, \ k \notin S_i\}$. We can then write $E_i = F \setminus S_i$, i.e. if agents in S_i are saved, then the remaining agents, not saved in F, are eliminated from the network. Similarly, define $E_j = F \setminus S_j$. Note next that when $S_i \in \mathcal{S}(s)$ and $S_j \notin \mathcal{S}(s)$, then $\tilde{E}_j \succsim \tilde{E}_i$. From the first part of Lemma 3 it follows directly that then $\overline{\mathbf{g}}^{-\tilde{E}_i} \supseteq \overline{\mathbf{g}}^{-\tilde{E}_j}$. That is, if $S_i \in \mathcal{S}(s)$ and $S_j \notin \mathcal{S}(s)$, then $\overline{\mathbf{g}}^{-F \setminus S_i} \supseteq \overline{\mathbf{g}}^{-F \setminus S_j}$. Assume that $\overline{\mathbf{g}}$ is not empty.[21] We can then use the same arguments as the ones used in Proposition 2 to show that $\mathcal{S}(s)$ is an optimal targeting policy.

Corollary to Proposition 2: *Assume $\overline{\mathbf{g}} \notin \{\overline{\mathbf{g}}^e\}$. If γ is sufficiently small, then saving a set of failing agents $S_i \in \mathcal{S}(s)$ is an optimal targeting policy for any value of δ.*

3.2 Large Competition/Congestion Effects

We next analyze optimal targeting when competition or congestion effects may be considered large. Prescriptions for a network that adapts may now differ from the case when the network does not adapt (even in the absence of targeting cost). The reason is that the elimination of a most central agent, who therefore also displays the highest effort level, decreases congestion by the most. The remaining agents may create new links in subsequent time periods and a denser network in later periods may outweigh the effect of an initially sparse network.

[20] To see this, assume to the contrary that the adjustment process converges to a network $\overline{\mathbf{g}}$ that is not a nested split graph. We know that no agent has an incentive to create a link (by the arguments provided above) and no agent finds it profitable to delete any subset of links (since we assume that the adjustment process converged). Then $\overline{\mathbf{g}}$ is a pairwise Nash equilibrium network that is not a nested split graph, which contradicts Proposition 1.

[21] Note that we disregard $\overline{\mathbf{g}}^e$ for similar reasons as in Proposition 2.

The following example illustrates some of the intricacies involved with this case. We show that for the same parameter values and two networks, which display the same type of network structure, different targeting policies are optimal. More precisely, assume $e = 1$ and consider two dominant group networks with different core sizes. In one case optimal targeting prescribes eliminating an agent in the core, thereby coinciding with the key player policy when the network is fixed. In the other case optimal targeting prescribes eliminating an agent in the periphery, thereby differing from the optimal targeting policy when the network is fixed.[22]

Example 2: Assume that $n = 8$, $\alpha = 1$, $\beta = 1$, $\lambda = 0.125$, $\gamma = 0.124$, $\kappa = 0.045$ and δ is sufficiently large. Consider two dominant group networks: $\bar{\mathbf{g}}$ is such that the size of the core is 5 and $\hat{\bar{\mathbf{g}}}$ such that the size of the core is 6. One can show that both, $\bar{\mathbf{g}}$ and $\hat{\bar{\mathbf{g}}}$, are pairwise Nash equilibrium networks. For $\bar{\mathbf{g}}$, the configuration after eliminating a peripheral agent, $(\mathbf{x}(\bar{\mathbf{g}}_0^{-p}), \bar{\mathbf{g}}_0^{-p})$, is also a pairwise Nash equilibrium, so that $\bar{\mathbf{g}}_t^{-p} = \bar{\mathbf{g}}_0^{-p}$ $\forall t \geq 0$. In contrast, in the configuration after eliminating a core agent in $\bar{\mathbf{g}}$, $(\mathbf{x}(\bar{\mathbf{g}}_0^{-c}), \bar{\mathbf{g}}_0^{-c})$, all agents not connected find it profitable to create a link and the network stays complete thereafter. That is, $\bar{\mathbf{g}}_t^{-c} = \bar{\mathbf{g}}^c$ $\forall t \geq 1$. For δ sufficiently large, optimal targeting then prescribes eliminating an agent in the periphery for $\bar{\mathbf{g}}$. In the case of $\hat{\bar{\mathbf{g}}}$, however, both $(\mathbf{x}(\hat{\bar{\mathbf{g}}}_0^{-p}), \hat{\bar{\mathbf{g}}}_0^{-p})$ and $(\mathbf{x}(\hat{\bar{\mathbf{g}}}_0^{-c}), \hat{\bar{\mathbf{g}}}_0^{-c})$ are pairwise Nash equilibria. Since $\hat{\bar{\mathbf{g}}}^{-c} \subset \hat{\bar{\mathbf{g}}}^{-p}$ holds by Lemma 3, it is optimal to target an agent in the core.

Below we provide sufficient conditions for a pairwise Nash equilibrium to exist such that the optimal targeting policy with an adaptive network differs from the fixed network case. The proof, together with a more detailed description, is provided in the online appendix. As in Ballester et al. (2006) we assume that $e = 1$. Assume further that $\lambda - \gamma \geq 0$ is sufficiently small and that δ is sufficiently large. There are then bounds on linking cost such that a pairwise Nash equilibrium exists in which it is optimal to eliminate a least central agent (with the fewest links and lowest inter-centrality). The intuition is again that eliminating a more central agent diminishes congestion effects more. This leads to the creation of new links and denser networks with higher aggregate effort levels in later time periods.

Proposition 3: *Assume $e = 1$. If $\lambda - \gamma \geq 0$ is sufficiently small, δ is sufficiently large and $\kappa_1' < \kappa < \kappa_2'$, then there exists a pairwise Nash equilibrium, such that the optimal targeting policy prescribes eliminating an agent with the fewest links.*

In the following we present the simplest case, a star network, that allows us to highlight the issues that may arise when γ is large. For simplicity we assume that $\lambda = \gamma$ and that $e = 1$ or $e = 2$. However, from our analysis it is clear that analogous results can be obtained for γ

[22] A network $\bar{\mathbf{g}}$ is a core-periphery network if the set of agents N can be partitioned into two sets, $C(\bar{\mathbf{g}})$ (the core) and $P(\bar{\mathbf{g}})$ (the periphery), such that $\bar{g}_{i,j} = 1$ $\forall i, j \in C(\bar{\mathbf{g}})$ and $\bar{g}_{i,j} = 0$ $\forall i, j \in P(\bar{\mathbf{g}})$. A group dominant network is a core-periphery network such that $\bar{g}_{i,j} = 0$ $\forall i, j : \ i \in C(\bar{\mathbf{g}}),\ j \in P(\bar{\mathbf{g}})$.

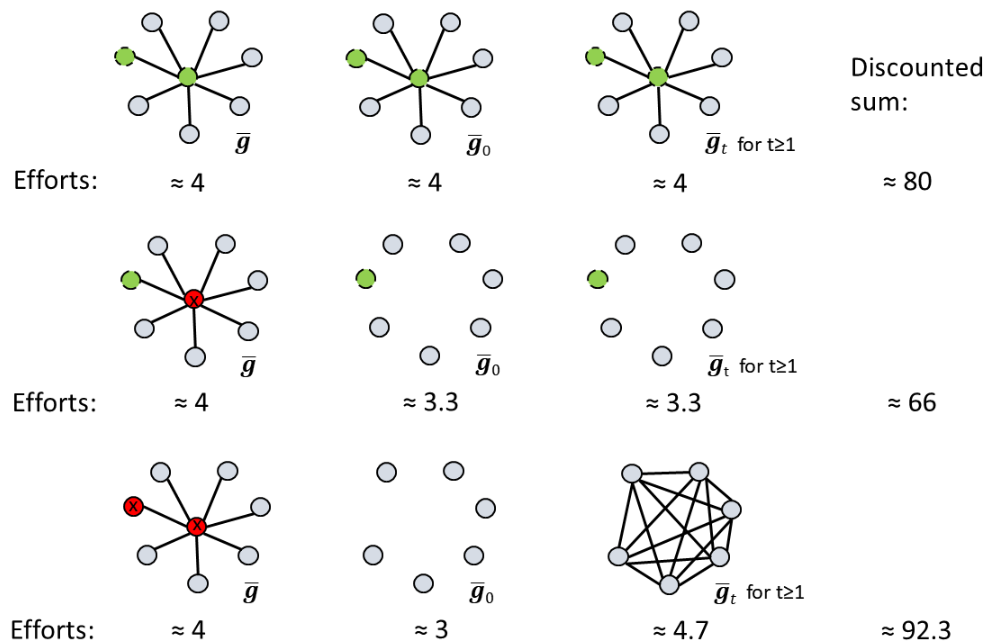
sufficiently close to λ. Moreover, conditions for $e > 2$ can easily be derived. We show that optimal targeting policies differ from the fixed network case and, in particular, that it may be optimal to not eliminate the maximum number of agents. Furthermore, following an optimal targeting policy under the assumption that the network is fixed, when in fact it adapts, may lead to a worse outcome than not intervening at all.

Denote by κ^e_{n-x} the marginal payoffs of two agents creating a link in the empty network of size $n-x$. Note that if $\kappa < \kappa^e_{n-x}$, then any pair of agents finds it profitable to create a link in the empty network of size $n-x$, while if $\kappa \geq \kappa^e_{n-x}$, then no pair of agents finds it profitable to create a link in the empty network of size $n-x$. Denote by $\underline{\kappa}^s_n$ and $\overline{\kappa}^s_n$ the relevant bounds on κ such that for $\kappa \in [\underline{\kappa}^s_n, \overline{\kappa}^s_n]$ the star network of n agents is a pairwise Nash equilibrium. All relevant linking cost are formally defined in the appendix (Definition 8). As it turns out, some of the conditions are simpler for $n \geq 8$ and we adopt this assumption throughout Proposition 4.

Proposition 4: *Assume* $\overline{\mathbf{g}}$ *is a pairwise Nash equilibrium,* δ *is sufficiently large and* $\lambda = \gamma$, *then the optimal targeting policy prescribes:*
i) if $e = 1$, *eliminate the central agent;*
ii) if $e = 2$ *and* $\kappa \in [\kappa^e_{n-2}, \overline{\kappa}^s_n]$, *eliminate the center and a peripheral agent;*
if $e = 2$, $\kappa \in [\underline{\kappa}^s_n, \kappa^e_{n-2})$ *and* $\lambda < \beta/(n^2-4n+3)$, *eliminate the central and a peripheral agent;*
if $e = 2$, $\kappa \in [\underline{\kappa}^s_n, \kappa^e_{n-2})$ *and* $\lambda \geq \beta/(n^2-4n+3)$, *eliminate the central agent.*

We briefly describe the results provided in Proposition 4. Assume first that $e = 1$. One can then show that $\kappa^e_{n-1} = \underline{\kappa}^s_n$ and therefore no new links are created after eliminating the central agent. The network is then empty and remains empty for all future time periods (Lemma 13). In contrast, the network is a star of size $n-1$ in time period zero when eliminating a peripheral agent. Note that the empty network of $n-1$ agents is the network with the lowest aggregate effort levels for $n-1$ agents and that aggregate effort levels are also lower than in a star of n agents. It is therefore optimal to eliminate the central agent. If $e = 2$ and $\kappa \in [\kappa^e_{n-2}, \overline{\kappa}^s_n]$, then the network stays empty after eliminating the central and a peripheral agent. The optimal targeting policy therefore prescribes eliminating the central and a peripheral agent by the arguments described above. Assume next that $e = 2$ and $\kappa \in [\underline{\kappa}^s_n, \kappa^e_{n-2})$. In this case, when eliminating the central and a peripheral agent, there are $n-2$ agents and the network is empty in $t = 0$ and complete for $t \geq 1$. When eliminating the central agent only, there are $n-1$ agents and the network is empty for $t \geq 0$. The aggregate effort level of the complete network of $n-2$ agents is strictly smaller than the aggregate effort level of the empty network of $n-1$ agents if and only if $\lambda < \beta/(n^2-4n+3)$. Therefore, if $\lambda < \beta/(n^2-4n+3)$ and δ is sufficiently large, it is optimal to eliminate the central and a peripheral agent. If $\lambda \geq \beta/(n^2-4n+3)$, then optimal targeting prescribes eliminating the central agent only.

Note that in the last case of Proposition 4, although two agents can be eliminated, it is optimal to target only one agent. This stands in contrast with the optimal targeting policy when the network is fixed, which prescribes eliminating the central and a peripheral agent, yielding an empty network of $n-2$ agents. Moreover, following the optimal targeting policy for a fixed network, when in fact the network adapts, may then be worse than not intervening at all. A general condition can be derived.[23] We provide an example below. Note that these results are not only relevant when γ is equal to λ and one can construct corresponding examples for $\lambda > \gamma > 0$.[24]



Example 3: Assume that $e = 2$, $n = 8$, $\alpha = 7$, $\beta = 8$, $\lambda = 1$, $\gamma = 1$, $\kappa = 0.25$ and δ is sufficiently large. From the conditions in Proposition 4 it follows that the optimal targeting policy prescribes eliminating the central agent only, which yields the empty network of $n-1$ agents. The aggregate effort level for this network is approximately 3.3. The optimal targeting policy when the network is fixed is to eliminate the central and a peripheral agent. This policy leads to an initially empty network of $n-2$ agents and an aggregate effort level of 3. However, when the network adapts, then the network is complete from time period $t = 1$ onward, yielding an aggregate effort level of approximately 4.7. In comparison, the aggregate effort level in a star of n agents displays an aggregate effort level of approximately 4. That is, for δ sufficiently large, applying the optimal targeting policy of a fixed network when in fact the network adapts, may lead to worse outcomes than no intervention.

[23] For $n \geq 8$ we need, in addition to the above conditions, that $\beta < (9 + \sum_{i=5}^{i=n-3} i)\lambda$ holds.

[24] For an example analogous to the one above, assume that $e = 2$, $n = 8$, $\alpha = \frac{9}{2}$, $\beta = \frac{59}{8}$, $\lambda = 1$, $\gamma = \frac{3}{4}$, $\kappa = \frac{5}{32}$ and δ is sufficiently large. One can show that the star of n agents is a pairwise Nash equilibrium and linking behavior coincides with the last case of Proposition 4. The (rounded) aggregate effort levels are given by, the empty network of $n-1$ agents: 2.5; the empty network of $n-2$ agents: 2.3; the complete network of $n-2$ agents: 3.9; and the star of n agents: 3.2.

4 Discussion

In the following we briefly comment on targeting policies for the applications that are best captured by our model: crime networks and R&D collaborations of firms.[25] The need for targeting in the context of crime arises from the high cost of crime control and incarceration, as well as large prison populations. For example, in the U.S. yearly expenditure on crime control amounts to approximately $270 billion, of which more than $80 billion is spent on incarceration alone, with currently 2.2 million imprisoned individuals nationwide.[26] Empirical studies have shown that targeting policies can be fruitfully applied to crime networks, providing more effective means to combat crime and that, in particular, these policies outperform traditional approaches.[27] At this point it is worthwhile to briefly discuss the implementability of targeting policies in the context of crime. One requirement is knowledge of the network and it may appear that this is difficult to obtain in the case of crime networks. However, such data exists or can often be retrieved.[28] Once a criminal network is mapped, there are different approaches to implementing a targeting policy in practice. While criminals are not to be imprisoned unless proven guilty of a crime, targeted criminals may be offered incentives to leave the network. This can be accomplished through heightened monitoring, providing job opportunities, employment and educational training, or even organizing geographic relocation. Policies of this sort have already been implemented for some time in the U.S. and Canada (see Tremblay et al., 1996).

Targeting is also relevant when deciding which subset of failing agents to save. A prominent example is the decision to save General Motors by the U.S. government in 2008. More generally, consider firms that engage in bilateral R&D partnerships, so as to benefit from cost-reducing technology spillovers, and compete in product markets.[29] These type of R&D partnerships are a common feature in many industries, particularly in those with rapid technological change. Examples include the pharmaceutical, chemical, computer and automotive industry.[30] Data

[25]Note that trade, as well as interbank lending networks, are typically thought of as directed and weighted and therefore a model featuring directed and weighted links is better suited for these applications. However, the adjustment process and more generally the approach presented here will be useful when studying these applications as well.

[26]See https://obamawhitehouse.archives.gov/sites/default/files/page/files/20160423_cea_incarceration_criminal_justice.pdf

[27]See Lindquist and Zenou (2014).

[28]For example, Sarnecki (2001) constructs a criminal network in Sweden by using police records, which register each time two (or more) individuals are suspected of a crime. Similar data is available in many countries. Tayebi et al. (2011) use a data set provided by the Royal Canadian Mounted Police (RCMP), which comprises five years of arrest-data and is available for research purposes. Coplink (Hauck et al., 2002) is a large scale research project in crime data mining in the United States. It uses information from various sources, such as habits of criminals and close associations in crime to capture network connections. Mastrobuoni and Patacchini (2012) use a data set from the Federal Bureau of Narcotics on U.S. mafia members, which allows the authors to construct a criminal network.

[29]See D'Aspremont and Jacquemin (1988) and Goyal and Moraga-Gonzalez (2001).

[30]See, for example Ahuja (2000), Hagedoorn (2002), Powell et al. (2005), Riccaboni and Pammolli (2002), Roijakkers and Hagedoorn (2006), König et al. (2014).

for R&D collaborations is, of course, readily available (e.g., via CATI and Compustat). An interesting study regarding the chemical and pharmaceutical sector is Hsie et al. (2018). The authors present a structural model and run counterfactual simulations regarding the exit of a single firm. They find that the most relevant firm, Pfizer Inc., displays the highest number of links.

5 Conclusion

This paper studies optimal targeting policies when agents may not only adjust effort levels, but also linking decisions after a set of agents is eliminated. A simple and tractable adjustment process is introduced and the following planner problem is defined: eliminate a set of up to e agents from an equilibrium network, such that the discounted aggregate effort level is minimal. We highlight the role of competition and congestion effects for targeting policies in settings where the network may adapt after agents were targeted. More precisely, if the parameter governing global strategic substitutes is sufficiently small, then the optimal targeting policy when the network adapts coincides with the fixed network case. This translates into a particularly simple policy for the equilibrium networks considered: remove the (sets of) agent(s) with the highest number of links (i.e., the most central agents). However, if the parameter governing global strategic substitutes is large, then allowing for an endogenous and adaptive network may overturn optimal targeting results for fixed networks. We provide conditions such that it may be optimal to target less central agents and such that it may be optimal to not eliminate the maximum number of agents. Moreover, we show that applying the optimal targeting policy of a fixed network to one that is, in fact, adaptive, may result in worse outcomes than not intervening at all.

6 Appendix

Derivation of payoff function: Crime

Before defining pairwise Nash equilibrium, the above payoff function is derived, based on Jackson and Zenou (2014) in the context of crime. Assume that expected gains of crime to agent i are given by

$$\pi_i(\mathbf{x}, \overline{\mathbf{g}}) = b_i(\mathbf{x}) - p_i(\mathbf{x}, \overline{\mathbf{g}})f,$$

with

$$\begin{cases} b_i(\mathbf{x}) = \alpha' x_i - \frac{1}{2}(\beta - \gamma)x_i^2 - \gamma x_i \sum_{j \in N} x_j \\ p_i(\mathbf{x}, \overline{\mathbf{g}}) = p_0 x_i (A - \lambda' \sum_{j \in N_i(\overline{\mathbf{g}})} x_j). \end{cases}$$

Expected cost of criminal activity, $p_i(\mathbf{x}, \overline{\mathbf{g}})f$, increases in own criminal activity, x_i, since being involved in more criminal activities increases the chance of being caught. Local strategic complementarities stem from a decrease in the apprehension probability in direct neighbors' involvement in crime, due to a direct know-how transfer. Note that A is assumed to be sufficiently large, so that the apprehension probability is always positive for all criminals.[31] Finally, global strategic substitutes are due to congestion effects for crime opportunities, captured by $\gamma x_i \sum_{j \in N} x_j$ in the expression for $b_i(\mathbf{x})$.[32]
Direct substitution yields

$$\pi_i(\mathbf{x}, \overline{\mathbf{g}}) = (\alpha' - p_0 f A)x_i - \frac{1}{2}\beta x_i^2 + p_0 f \lambda' x_i \sum_{j \in N_i(\overline{\mathbf{g}})} x_j - \gamma x_i \sum_{j \in N \setminus \{i\}} x_j.$$

For $\alpha = \alpha' - p_0 f A > 0$ and $\lambda = p_0 f \lambda'$ these payoffs are equivalent to the specification used in Ballester et al. (2006).

Derivation of payoff function: Research and Development

We present here the arguably simplest derivation of the payoff specification in Ballester et al. (2006) in the context of R&D collaborations. For alternative derivations which include R&D efforts and explicitly model consumers and multiple markets see, for example, König (2016). Firms may enter into R&D collaborations, which cause knowledge spillovers due to learning-by-doing effects. Cost reduction depends on a firm's own production level and on the production level of collaborating firms. Given production level q_i, marginal cost of firm i, c_i, are given by

[31] See König, Liu and Zenou (2014) for how to calculate an appropriate lower bound on A.

[32] One way to argue for as to why congestion effects should affect agents with higher criminal activity more, as reflected in the term $\gamma x_i \sum_{j \in N} x_j$, is that when aggregate crime levels are higher, the public may become more vigilant, which in turn has a higher impact on agents with high individual levels of criminal activity.

$$c_i = \bar{c} - \mu q_i - \lambda \sum_{j \in N_i(\bar{\mathbf{g}})} q_j$$

Firm i's profits are given by

$$\pi_i = p_i q_i - c_i q_i.$$

Assume inverse demand for the good q_i is given by $p_i = a - bq_i - \gamma \sum_{j \neq i} q_j$. Substituting into firm i's profits we obtain

$$\pi_i = (a_i - bq_i - \gamma \sum_{j \neq i} q_j)q_i - (\bar{c}_i - gq_i - \lambda \sum_{j \in N_i(\bar{\mathbf{g}})} q_j)q_i.$$

Collecting terms yields

$$\pi_i = (a - \bar{c})q_i - (b - g)q_i^2 + \lambda q_i \sum_{j \in N_i(\bar{\mathbf{g}})} q_j - \gamma q_i \sum_{j \neq i} q_j.$$

Setting a, $\bar{c}$, b and g such that $(a - \bar{c}) = \alpha$, $(b - g) = \beta + \gamma$ then yields the specification used in Ballester et al. (2006).

Definition 5: (Mahadev and Peled, 1995). Let $\bar{\mathbf{g}}$ be a graph whose distinct positive degrees are $\eta_{(1)}(\bar{\mathbf{g}}) < \eta_{(2)}(\bar{\mathbf{g}}) < ... < \eta_{(k)}(\bar{\mathbf{g}})$ and let $d_0(\bar{\mathbf{g}}) = 0$ (even if no agent with degree 0 exists in $\bar{\mathbf{g}}$). Define $\mathcal{D}_j(\bar{\mathbf{g}}) = \{i \in N : \eta_i(\bar{\mathbf{g}}) = \eta_{(j)}(\bar{\mathbf{g}})\}$ for $j = 0, ..., k$. Then the set-valued vector $\mathcal{D}(\bar{\mathbf{g}}) = (\mathcal{D}_0(\bar{\mathbf{g}}), \mathcal{D}_1(\bar{\mathbf{g}}), ..., \mathcal{D}_k(\bar{\mathbf{g}}))$ is called the degree partition of $\bar{\mathbf{g}}$.

Definition 6: (Mahadev and Peled, 1995). Consider a nested split graph $\bar{\mathbf{g}}$ and let $\mathcal{D}(\bar{\mathbf{g}}) = (\mathcal{D}_0(\bar{\mathbf{g}}), \mathcal{D}_1(\bar{\mathbf{g}}), ..., \mathcal{D}_k(\bar{\mathbf{g}}))$ be its degree partition. Then the nodes N can be partitioned in independent sets $\mathcal{D}_j$, $j = 1, ..., \lfloor k/2 \rfloor$ and a dominating set $\cup_{j=\lfloor k/2 \rfloor+1}^{k} \mathcal{D}_j$ in the graph $\bar{\mathbf{g}}^{-\mathcal{D}_0}$.[33] Moreover, the neighborhoods are nested. In particular, for each node $i \in \mathcal{D}_j$, $j = 1, ..., k$,

$$N_i(\bar{\mathbf{g}}) = \begin{cases} \cup_{l=1}^{j} \mathcal{D}_{k+1-l}(\bar{\mathbf{g}}) & if\ j = 1, ..., \lfloor k/2 \rfloor \\ \cup_{l=1}^{j} \mathcal{D}_{k+1-l}(\bar{\mathbf{g}}) \setminus \{i\} & if\ j = \lfloor k/2 \rfloor + 1, ..., k. \end{cases}$$

To simplify notation we denote $I(\bar{\mathbf{g}}) = \cup_{j=1}^{\lfloor k/2 \rfloor} \mathcal{D}_j$ and $D(\bar{\mathbf{g}}) = \cup_{j=\lfloor k/2 \rfloor+1}^{k} \mathcal{D}_j$. Note that when $\bar{\mathbf{g}}$ is empty, then in the graph $\bar{\mathbf{g}}^{-\mathcal{D}_0}$ the set of agents $N \setminus D_0$ is empty and therefore $D(\bar{\mathbf{g}})$ is empty. Next we define a threshold, $t(\bar{\mathbf{g}})$, which depends on the degree partition as follows.

Definition 7: $t(\bar{\mathbf{g}}) = \begin{cases} \mid D(\bar{\mathbf{g}}) \mid & if\ k\ is\ even \\ \mid D(\bar{\mathbf{g}}) \mid -1 & if\ k\ is\ odd \end{cases}$

[33] A dominating set for a graph is a subset D of N such that every node not in D is adjacent to at least one node in D. An independent set for a graph is a subset I of N such that for every two nodes in I, there is no link between the two.

Lemma 1: *Assume* $\bar{\mathbf{g}}$ *is a nested split graph such that* $\bar{\mathbf{g}} \notin \{\bar{\mathbf{g}}^e, \bar{\mathbf{g}}^c\}$. *There exists a set* E *such that* $\bar{\mathbf{g}}^{-E}$ *is the empty network if and only if* $e = \mid E \mid \geq t(\bar{\mathbf{g}})$. *Moreover,*

i) if k *is even, then* $\bar{\mathbf{g}}^{-E}$ *is the empty network if and only if* E *is such that either*

$D(\bar{\mathbf{g}}) \subseteq E$ *or*

$\{\{D(\bar{\mathbf{g}}) \setminus m\} \cup N_m(\bar{\mathbf{g}})\} \subseteq E$ *and* $m \in D(\bar{\mathbf{g}})$*;*

ii) if k *is odd, then* $\bar{\mathbf{g}}^{-E}$ *is the empty network if and only if* E *is such that either*

$\{D(\bar{\mathbf{g}}) \setminus m\} \subseteq E$ *and* $m \in \mathcal{D}_{\lfloor \frac{k}{2} \rfloor + 1}(\bar{\mathbf{g}})$ *or*

$\{\{D(\bar{\mathbf{g}}) \setminus m\} \cup N_m(\bar{\mathbf{g}})\} \subseteq E$ *and* $m \in D(\bar{\mathbf{g}}) \setminus \mathcal{D}_{\lfloor \frac{k}{2} \rfloor + 1}(\bar{\mathbf{g}})$.

Proof. We first consider the case when k is even. If $k = 0$ then the dominating set $D(\bar{\mathbf{g}})$ is the empty set and $\bar{\mathbf{g}}$ is empty. Assume therefore that $k \geq 2$. Assume first that $D(\bar{\mathbf{g}}) \subseteq E$. To see that then $\bar{\mathbf{g}}^{-E}$ is empty, note that each of the remaining agents in $\bar{\mathbf{g}}^{-E}$ is either a singleton agent in $\bar{\mathbf{g}}$ or in some independent set $\mathcal{D}_j$ with $j = 1, ..., \lfloor k/2 \rfloor$ in $\bar{\mathbf{g}}$. From Definition 6 it follows directly that there are no links among any of these agents in $\bar{\mathbf{g}}$ and therefore $\bar{\mathbf{g}}^{-E}$ is empty. Next we show that if k is even, $\{\{D(\bar{\mathbf{g}}) \setminus m\} \cup N_m(\bar{\mathbf{g}})\} \subseteq E$ and $m \in D(\bar{\mathbf{g}})$, then $\bar{\mathbf{g}}^{-E}$ is the empty network. If $m \in E$, then $D(\bar{\mathbf{g}}) \subseteq E$ also holds and $\bar{\mathbf{g}}^{-E}$ is empty by the previous argument. Assume next that $m \notin E$. Since $N_m(\bar{\mathbf{g}}) \subseteq E$, m is a singleton in $\bar{\mathbf{g}}^{-E}$. Note also that, since $\{D(\bar{\mathbf{g}}) \setminus m\} \subseteq E$, all agents in $\bar{\mathbf{g}}^{-E}$ other than m are in some independent set $\mathcal{D}_j$ for $j = 1, ..., \lfloor k/2 \rfloor$ in $\bar{\mathbf{g}}$, or singletons in $\bar{\mathbf{g}}$. From Definition 6 it follows that there are then no links among any of these agents in $\bar{\mathbf{g}}$ and therefore $\bar{\mathbf{g}}^{-E}$ is empty. We next show by contraposition that, if $\bar{\mathbf{g}}^{-E}$ is empty, then E must satisfy one of the two (or both) conditions in i). Assume therefore that $\bar{\mathbf{g}}^{-E}$ is not empty. Note that from Definition 6 we know that there are only two types of links. Those between agents i, j with $i, j \in D(\bar{\mathbf{g}})$, and links between i, j such that $i \in D(\bar{\mathbf{g}})$ and $j \in I(\bar{\mathbf{g}})$. Note that from Definition 6 we also know that, since $k \geq 2$ and k is even, that each agent in $D(\bar{\mathbf{g}})$ is linked to some agent in $I(\bar{\mathbf{g}})$ and, since we assume in i) that $m \in D(\bar{\mathbf{g}})$, we know that $N_m(\bar{\mathbf{g}}) \setminus D(\bar{\mathbf{g}}) \neq \emptyset$. Therefore, for $\bar{\mathbf{g}}^{-E}$ to not be empty, there either exists a pair of agents $i, j \in D(\bar{\mathbf{g}})$ such that $i, j \notin E$, or there exists a pair of agents $m \in D(\bar{\mathbf{g}})$ and $i \in N_m(\bar{\mathbf{g}}) \setminus D(\bar{\mathbf{g}})$ such that $m, i \notin E$. But then either $D(\bar{\mathbf{g}}) \subseteq E$, or $\{\{D(\bar{\mathbf{g}}) \setminus m\} \cup N_m(\bar{\mathbf{g}})\} \subseteq E$ and $m \in D(\bar{\mathbf{g}})$ does not hold. Next we show that there exists a set E such that $\bar{\mathbf{g}}^{-E}$ is empty if and only if $e \geq t(\bar{\mathbf{g}})$. Recall that if k is even and $k \geq 2$, then by Definition 6 each agent in $D(\bar{\mathbf{g}})$ is connected to at least one agent not in $D(\bar{\mathbf{g}})$. Therefore, $\mid \{\{D(\bar{\mathbf{g}}) \setminus m\} \cup N_m(\bar{\mathbf{g}})\} \mid \geq \mid D(\bar{\mathbf{g}}) \mid$. That is, if $e < t(\bar{\mathbf{g}}) = \mid D(\bar{\mathbf{g}}) \mid$ neither $D(\bar{\mathbf{g}}) \subseteq E$, nor $\{\{D(\bar{\mathbf{g}}) \setminus m\} \cup N_m(\bar{\mathbf{g}})\} \subseteq E$ and $m \in D(\bar{\mathbf{g}})$ can hold. If, however, $e \geq t(\bar{\mathbf{g}})$, then $\mid E \mid > \mid D(\bar{\mathbf{g}}) \mid$ and for any set E such that $D(\bar{\mathbf{g}}) \subseteq E$, the resulting network $\bar{\mathbf{g}}^{-E}$ is empty. That is, there exits a set E such that $\bar{\mathbf{g}}^{-E}$ is empty if and only if $e \geq t(\bar{\mathbf{g}})$. We consider next the case when k is odd. Assume first that $\{D(\bar{\mathbf{g}}) \setminus m\} \subseteq E$ and $m \in \mathcal{D}_{\lfloor \frac{k}{2} \rfloor + 1}(\bar{\mathbf{g}})$. To see that then $\bar{\mathbf{g}}^{-E}$ is empty, note that if k is odd, then from Definition 6 we know that agents in $\mathcal{D}_{\lfloor \frac{k}{2} \rfloor + 1}(\bar{\mathbf{g}})$

only sustain links to agents in $D(\bar{\mathbf{g}})$. Therefore, if $m \in \mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\bar{\mathbf{g}})$ and $\{D(\bar{\mathbf{g}}) \setminus m\} \subseteq E$, then m is a singleton agent in $\bar{\mathbf{g}}^{-E}$. Each of the remaining agents in $\bar{\mathbf{g}}^{-E}$ is either a singleton agent in $\bar{\mathbf{g}}$ or is in some independent set $\mathcal{D}_j$ for $j = 1, ..., \lfloor k/2 \rfloor$ in $\bar{\mathbf{g}}$. From Definition 6 it follows that there are no links among any of these agents in $\bar{\mathbf{g}}$ and therefore $\bar{\mathbf{g}}^{-E}$ is empty. The case for $\{\{D(\bar{\mathbf{g}}) \setminus m\} \cup N_m(\bar{\mathbf{g}})\} \subseteq E$ and $m \in D(\bar{\mathbf{g}}) \setminus \mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\bar{\mathbf{g}})$ is analogous to the case when k is even and E is such that $\{\{D(\bar{\mathbf{g}}) \setminus m\} \cup N_m(\bar{\mathbf{g}})\} \subseteq E$ and $m \in D(\bar{\mathbf{g}})$ and is omitted. Next we show by contraposition that if $\bar{\mathbf{g}}^{-E}$ is empty, then E must satisfy one of the two conditions (or both) in ii). Assume therefore that $\bar{\mathbf{g}}^{-E}$ is not empty. Note that from Definition 6 we know that there are only two types of links. Those between agents i, j with $i, j \in D(\bar{\mathbf{g}})$, and links between i, j such that $i \in D(\bar{\mathbf{g}})$ and $j \in I(\bar{\mathbf{g}})$. Note further that from Definition 6 we also know that each agent in $D(\bar{\mathbf{g}})$ is linked to some agent in $I(\bar{\mathbf{g}})$, with the exception of agents in $\mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\bar{\mathbf{g}})$, who are only linked to agents in $D(\bar{\mathbf{g}})$. Therefore, for $\bar{\mathbf{g}}^{-E}$ to not be empty, there either exist two agents $i, j \in D(\bar{\mathbf{g}})$ such that $i, j \notin E$, or there exists a pair of agents $m \in D(\bar{\mathbf{g}}) \setminus \mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\bar{\mathbf{g}})$ and $i \in N_m(\bar{\mathbf{g}}) \setminus D(\bar{\mathbf{g}})$ such that $m, i \notin E$. (Note that for an agent $m \in \mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\bar{\mathbf{g}})$, the set $N_m(\bar{\mathbf{g}}) \setminus D(\bar{\mathbf{g}})$ is the empty set). But then either $\{D(\bar{\mathbf{g}}) \setminus m\} \subseteq E$ and $m \in \mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\bar{\mathbf{g}})$, or $\{\{D(\bar{\mathbf{g}}) \setminus m\} \cup N_m(\bar{\mathbf{g}})\} \subseteq E$ and $m \in D(\bar{\mathbf{g}}) \setminus \mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\bar{\mathbf{g}})$ does not hold. Next we show for k odd that there exists a set E such that $\bar{\mathbf{g}}^{-E}$ is empty if and only if $e \geq t(\bar{\mathbf{g}})$. Since each agent in $D(\bar{\mathbf{g}}) \setminus \mathcal{D}_{\lfloor \frac{k}{2} \rfloor}(\bar{\mathbf{g}})$ is connected to at least one agent not in $D(\bar{\mathbf{g}})$, while agents in $\mathcal{D}_{\lfloor \frac{k}{2} \rfloor}(\bar{\mathbf{g}})$ are only connected to agents in $D(\bar{\mathbf{g}})$, we know that $\mid \{\{D(\bar{\mathbf{g}}) \setminus m\} \cup N_m(\bar{\mathbf{g}})\} \mid \geq \mid D(\bar{\mathbf{g}}) \mid -1$ holds. Therefore, if $e < t(\bar{\mathbf{g}}) = \mid D(\bar{\mathbf{g}}) \mid -1$, then neither $\{D(\bar{\mathbf{g}}) \setminus m\} \subseteq E$ and $m \in \mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\bar{\mathbf{g}})$ nor $\{\{D(\bar{\mathbf{g}}) \setminus m\} \cup N_m(\bar{\mathbf{g}})\} \subseteq E$ and $m \in D(\bar{\mathbf{g}}) \setminus \mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\bar{\mathbf{g}})$ can hold. If, however, $e \geq t(\bar{\mathbf{g}})$, then there exists a set E (with $\mid E \mid \geq t(\bar{\mathbf{g}})$) such that $\{D(\bar{\mathbf{g}}) \setminus m\} \subseteq E$ and $m \in \mathcal{D}_{\lfloor \frac{k}{2} \rfloor}(\bar{\mathbf{g}})$ and $\bar{\mathbf{g}}^{-E}$ is empty. That is, there exists a set E such that $\bar{\mathbf{g}}^{-E}$ is empty if and only if $e \geq t(\bar{\mathbf{g}})$. *Q.E.D.*

Lemma 2: Assume $\bar{\mathbf{g}}$ is a nested split graph. If k is even, then $\mid \mathcal{D}_{\lfloor \frac{k}{2} \rfloor}(\bar{\mathbf{g}}) \mid \geq 2$ and if k is odd, then $\mid \mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\bar{\mathbf{g}}) \mid \geq 2$.

Proof. Assume first that k is even. If $k = 0$, then $\bar{\mathbf{g}}$ is the empty network and $\mathcal{D}_{\lfloor \frac{k}{2} \rfloor}(\bar{\mathbf{g}}) = \mathcal{D}_0(\bar{\mathbf{g}}) = N$. Since $n \geq 3$, $\mid \mathcal{D}_{\lfloor \frac{k}{2} \rfloor}(\bar{\mathbf{g}}) \mid \geq 2$ holds. Assume next that k is even and $k \geq 2$. From Definition 6 we know that all agents in $\mathcal{D}_{\lfloor \frac{k}{2} \rfloor}(\bar{\mathbf{g}})$ are connected to all agents in the dominating set $D(\bar{\mathbf{g}})$, while agents in $\mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\bar{\mathbf{g}})$ are connected to all agents in the dominating set $D(\bar{\mathbf{g}})$ and to all agents in $\mathcal{D}_{\lfloor \frac{k}{2} \rfloor}(\bar{\mathbf{g}})$. Assume that $\mid \mathcal{D}_{\lfloor \frac{k}{2} \rfloor}(\bar{\mathbf{g}}) \mid = 1$. Then $\eta_i(\bar{\mathbf{g}}) = \eta_j(\bar{\mathbf{g}})$ for $i \in \mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\bar{\mathbf{g}})$ and $j \in \mathcal{D}_{\lfloor \frac{k}{2} \rfloor}(\bar{\mathbf{g}})$ and we have reached a contradiction. Assume next that k is odd. Assume first that $k = 1$ and, contrary to the above, that $\mid \mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\bar{\mathbf{g}}) \mid = 1$. But then $\mid \mathcal{D}_1(\bar{\mathbf{g}}) \mid = 1$ and agent $i \in \mid \mathcal{D}_1(\bar{\mathbf{g}}) \mid$ is therefore a singleton. That is, all agents are singletons, so that $k = 0$ and we have reached a contradiction. Assume next that k is odd and $k \geq 3$. Note that from Definition 6 we know that agents in the independent set $\mathcal{D}_{\lfloor \frac{k}{2} \rfloor}(\bar{\mathbf{g}})$ are linked to all agents in $D(\bar{\mathbf{g}}) \setminus \mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\bar{\mathbf{g}})$, but not to any agents in $\mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\bar{\mathbf{g}})$, while agents in $\mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\bar{\mathbf{g}})$ are linked to all agents in $D(\bar{\mathbf{g}})$ (and not to any agents in $I(\overline{\mathbf{g}})$). But then, if $\mid \mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\overline{\mathbf{g}}) \mid= 1$, then $\eta_i(\overline{\mathbf{g}}) = \eta_j(\overline{\mathbf{g}})$ for $i \in \mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\overline{\mathbf{g}})$ and $j \in \mathcal{D}_{\lfloor \frac{k}{2} \rfloor}(\overline{\mathbf{g}})$ and we have again reached a contradiction. *Q.E.D.*

Lemma 3: *Assume $\overline{\mathbf{g}}$ is a nested split graph, $\overline{\mathbf{g}} \notin \{\overline{\mathbf{g}}^e\}$ and $e < t(\overline{\mathbf{g}})$. If $E_i \in \mathscr{E}(e)$ and $E_j \notin \mathscr{E}(e)$, then $\overline{\mathbf{g}}^{-E_i} \subset \overline{\mathbf{g}}^{-E_j}$.*

Proof. Note first that if $E_i \in \mathscr{E}(e)$ and $E_j \notin \mathscr{E}(e)$, then it follows directly from the definition of $\mathscr{E}(e)$ that $\tilde{E}_i \succ \tilde{E}_j$ holds (and therefore $\tilde{E}_i \succsim \tilde{E}_j$). We first show that if $E_i \in \mathscr{E}(e)$ and $E_j \notin \mathscr{E}(e)$, then any link that is present in $\overline{\mathbf{g}}^{-E_i}$ is also present in $\overline{\mathbf{g}}^{-E_j}$. Since any agents in $E_i \cap E_j$ are eliminated in both E_i and E_j, we can disregard any links involving an agent in this set. Note next that, since linking behavior between pairs of agents $k, l \notin E_i \cup E_j$ remains unchanged and is the same in both $\overline{\mathbf{g}}^{-E_i}$ and $\overline{\mathbf{g}}^{-E_j}$, we can disregard links between these pairs of agents as well. Next consider the link between an agent that is neither eliminated nor relabeled in $\overline{\mathbf{g}}^{-E_i}$ and $\overline{\mathbf{g}}^{-E_j}$, and a relabeled agent in $\overline{\mathbf{g}}^{-E_i}$ and $\overline{\mathbf{g}}^{-E_j}$, respectively. That is, consider the link between an agent $k \notin E_i \cup E_j$ and agent $r_x(\overline{\mathbf{g}}^{-E_i})$ in $\overline{\mathbf{g}}^{-E_i}$, and the link between $k \notin E_i \cup E_j$ and agent $r_x(\overline{\mathbf{g}}^{-E_j})$ in $\overline{\mathbf{g}}^{-E_j}$. Recall that $\tilde{E}_1 \cap \tilde{E}_2 = \emptyset$ and, since $\tilde{E}_i \succsim \tilde{E}_j$, that $\eta^{(x,\tilde{E}_i)}(\overline{\mathbf{g}}) \geq \eta^{(x,\tilde{E}_j)}(\overline{\mathbf{g}})$ holds $\forall x \in \{1, ..., \tilde{e}\}$. Fix x, so that we can write $\eta_l^{(x,\tilde{E}_i)}(\overline{\mathbf{g}}) \geq \eta_m^{(x,\tilde{E}_j)}(\overline{\mathbf{g}})$. Note that then $r_x(\overline{\mathbf{g}}^{-E_i}) = m$, while $r_x(\overline{\mathbf{g}}^{-E_j}) = l$. Since $\overline{\mathbf{g}}$ is a nested split graph and $\eta_l^{(x,\tilde{E}_i)}(\overline{\mathbf{g}}) \geq \eta_m^{(x,\tilde{E}_j)}(\overline{\mathbf{g}})$ holds, we know that if $\overline{g}_{k,m} = 1$, then $\overline{g}_{k,l} = 1 \ \forall k \notin \{l, m\}$ in $\overline{\mathbf{g}}$. Therefore, if $\overline{g}^{-E_i}_{k,r_x(\overline{\mathbf{g}}^{-E_i})} = 1$ in $\overline{\mathbf{g}}^{-E_i}$, then $\overline{g}^{-E_j}_{k,r_x(\overline{\mathbf{g}}^{-E_j})} = 1$ in $\overline{\mathbf{g}}^{-E_j}$. Next, consider links between relabeled agents in $\overline{\mathbf{g}}^{-E_i}$ and $\overline{\mathbf{g}}^{-E_j}$, respectively. That is, assume $x \neq x'$ and consider the link between a pair of agents $r_x(\overline{\mathbf{g}}^{-E_i})$ and $r_{x'}(\overline{\mathbf{g}}^{-E_i})$ in $\overline{\mathbf{g}}^{-E_i}$ and the link between a pair of agents $r_x(\overline{\mathbf{g}}^{-E_j})$ and $r_{x'}(\overline{\mathbf{g}}^{-E_j})$ in $\overline{\mathbf{g}}^{-E_j}$. Fixing x and x', we know from $\tilde{E}_i \succsim \tilde{E}_j$ that $\eta_l^{(x,\tilde{E}_i)}(\overline{\mathbf{g}}) \geq \eta_m^{(x,\tilde{E}_j)}(\overline{\mathbf{g}})$ and $\eta_s^{(x',\tilde{E}_i)}(\overline{\mathbf{g}}) \geq \eta_t^{(x',\tilde{E}_j)}(\overline{\mathbf{g}})$. From $\tilde{E}_1 \cap \tilde{E}_2 = \emptyset$ we know that l, m, s, t are distinct. Note that $r_x(\overline{\mathbf{g}}^{-E_i}) = m$ and $r_{x'}(\overline{\mathbf{g}}^{-E_i}) = t$, while $r_x(\overline{\mathbf{g}}^{-E_j}) = l$ and $r_{x'}(\overline{\mathbf{g}}^{-E_j}) = s$. Again, since $\overline{\mathbf{g}}$ is a nested split graph and $\eta_l^{(x,\tilde{E}_i)}(\overline{\mathbf{g}}) \geq \eta_m^{(x,\tilde{E}_j)}(\overline{\mathbf{g}})$ and $\eta_s^{(x',\tilde{E}_i)}(\overline{\mathbf{g}}) \geq \eta_t^{(x',\tilde{E}_j)}(\overline{\mathbf{g}})$ holds, we know that if $\overline{g}^{-E_i}_{r_x(\overline{\mathbf{g}}),r_{x'}(\overline{\mathbf{g}}^{-E_i})} = 1$ in $\overline{\mathbf{g}}^{-E_i}$, then $\overline{g}^{-E_j}_{r_x(\overline{\mathbf{g}}^{-E_j}),r_{x'}(\overline{\mathbf{g}}^{-E_j})} = 1$ in $\overline{\mathbf{g}}^{-E_j}$. Therefore, all links that are present in $\overline{\mathbf{g}}^{-E_i}$ are also present in $\overline{\mathbf{g}}^{-E_j}$ and $\overline{\mathbf{g}}^{-E_i} \subseteq \overline{\mathbf{g}}^{-E_j}$. To show that $\overline{\mathbf{g}}^{-E_i} \subset \overline{\mathbf{g}}^{-E_j}$ holds, it is sufficient to show that the number of links that are deleted when eliminating E_i is strictly higher than when eliminating E_j from $\overline{\mathbf{g}}$. Note first that since $e < t(\overline{\mathbf{g}})$ and $E_i \in \mathscr{E}(e)$, we know that $E_i \subset D(\overline{\mathbf{g}})$. We distinguish two cases. Assume first that $E_j \subset D(\overline{\mathbf{g}})$. Note that all pairs of agents in $D(\overline{\mathbf{g}})$ are connected, and if there exists a pair of agents $l, m \in D(\overline{\mathbf{g}})$ such that $\eta_l(\overline{\mathbf{g}}) > \eta_m(\overline{\mathbf{g}})$, then there must exist at least one agent in $I(\overline{\mathbf{g}})$ to which l is connected, while m is not. Moreover, since $\overline{\mathbf{g}}$ is a nested split graph, $N_l(\overline{\mathbf{g}}) \setminus \{k\} \subset N_k(\overline{\mathbf{g}}) \setminus \{l\}$ holds. From $\tilde{E}_i \succ \tilde{E}_j$ we know that $\eta^{(x,E_i)}(\overline{\mathbf{g}}) \geq \eta^{(x,E_j)}(\overline{\mathbf{g}}) \ \forall x$ and there exists a x' such that $\eta_l^{(x',E_i)}(\overline{\mathbf{g}}) > \eta_m^{(x',E_j)}(\overline{\mathbf{g}})$. Therefore, for any such x', eliminating $l \in E_i$ deletes the same number of links to agents in $D(\overline{\mathbf{g}})$ as $m \in E_j$. Moreover, $l \in E_i$ deletes all links to agents to which $m \in E_j$ is connected and at least one link to an agent to which $m \in E_j$ is not connected in $I(\overline{\mathbf{g}})$. By the same argument all links that are deleted in E_j are also deleted in E_j (and possibly more) for all $x \in \{1, 2, ..., \tilde{e}\}$.

Therefore, a strictly higher number of links is deleted when eliminating E_i than when eliminating E_j. Next consider the case when $E_j \subset D(\overline{\mathbf{g}})$ does not hold. Assume first that there is a single agent $l \in E_j$ such that $l \in I(\overline{\mathbf{g}})$. We distinguish two cases. Assume $l \in \mathcal{D}_0(\bar{\mathbf{g}})$ (i.e., l is a singleton) and consider the set E'_j, given by $E'_j = \{E_j \setminus l\} \cup m$ with $m : m \in D(\overline{\mathbf{g}})$ and $m \notin E_j$. Note that such an agent m exists, since $e < t(\overline{\mathbf{g}})$. E'_j deletes a strictly higher number of links than E_j, since m is connected to all agents in $D(\overline{\mathbf{g}})$, while no links are deleted when eliminating the singleton agent l. Note further that $E'_j \subseteq D(\overline{\mathbf{g}})$. If $E'_j \in \mathscr{E}(e)$, then the same number of links are eliminated in E'_j as in E_i, since then $\tilde{E}_i \sim \tilde{E}_j$ and $\overline{\mathbf{g}}^{-E_i} = \overline{\mathbf{g}}^{-E_j}$. To see the latter, note that if $\tilde{E}_i \sim \tilde{E}_j$, then $\eta_l^{(x,\tilde{E}_i)}(\overline{\mathbf{g}}) = \eta_m^{(x,\tilde{E}_j)}(\overline{\mathbf{g}})$ for all $x \in \{1, 2, ..., \tilde{e}\}$. Since $\overline{\mathbf{g}}$ is a nested split graph, $\overline{g}_{k,m} = 1$ if and only if $\overline{g}_{k,l} = 1$ $\forall k \notin \{m, l\}$ and therefore $\overline{\mathbf{g}}^{-E_i} = \overline{\mathbf{g}}^{-E_j}$. If $E'_j \notin \mathscr{E}(e)$, then E_i deletes a strictly higher number of agents than E'_j, as shown above. Therefore, a strictly higher number of links are eliminated in E_i than in E_j. Assume next that there is a single agent $l \in E_j$ such that $l \in I(\overline{\mathbf{g}})$, but $l \notin \mathcal{D}_0(\bar{\mathbf{g}})$. We distinguish two subcases. Assume first that k is odd. From Definition 6 we know that $\mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\bar{\mathbf{g}}) \in D(\overline{\mathbf{g}})$ and that there are no links between agents in $\mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\bar{\mathbf{g}})$ and agents in $I(\bar{\mathbf{g}})$. From $| \mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\bar{\mathbf{g}}) | \geq 2$ (Lemma 2) it therefore follows that each agent in $I(\overline{\mathbf{g}})$ is not linked to at least two agents in $D(\overline{\mathbf{g}})$, while all agents in $D(\overline{\mathbf{g}})$ are connected among each other. Therefore, E'_j, given by $E'_j = \{E_j \setminus l\} \cup m$ with $m :\ m \in D(\overline{\mathbf{g}})$ and $m \notin E_j$, deletes a strictly higher number of links than E_j. To see this, note that $m \in D(\overline{\mathbf{g}})$ is connected to all agents to which l is connected. Moreover, m is connected to at least one agent in $\mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\bar{\mathbf{g}})$, while l is not connected to any agent in $\mathcal{D}_{\lfloor \frac{k}{2} \rfloor+1}(\bar{\mathbf{g}})$. Therefore, a strictly higher number of links is deleted in E'_j than in E_j. Since $E'_j \subset D(\overline{\mathbf{g}})$ we know by the above argument that a strictly higher number of links is deleted when eliminating E_i than E_j. Assume next that k is even, $l \in I(\overline{\mathbf{g}})$, but $l \notin \mathcal{D}_0(\bar{\mathbf{g}})$. Note that from Definition 6 we know that the set $\mathcal{D}_{\lfloor \frac{k}{2} \rfloor}(\bar{\mathbf{g}}) \in I(\overline{\mathbf{g}})$, that each agent in $D(\overline{\mathbf{g}})$ is linked to all agents in $\mathcal{D}_{\lfloor \frac{k}{2} \rfloor}(\bar{\mathbf{g}})$, and that $| \mathcal{D}_{\lfloor \frac{k}{2} \rfloor}(\bar{\mathbf{g}}) | \geq 2$ (by Lemma 2). Again the set E'_j, given by $E'_j = \{E_j \setminus l\} \cup m$ with $m : m \in D(\overline{\mathbf{g}})$ and $m \notin E_j$, deletes a strictly higher number of links than E_j. To see this, note that $m \in D(\overline{\mathbf{g}})$ is connected to all agents that are linked to l. Moreover, since $| \mathcal{D}_{\lfloor \frac{k}{2} \rfloor}(\bar{\mathbf{g}}) | \geq 2$, $m \in D(\overline{\mathbf{g}})$ is connected to all agents in $\mathcal{D}_{\lfloor \frac{k}{2} \rfloor}(\bar{\mathbf{g}})$, while l is not connected to any agent in $\mathcal{D}_{\lfloor \frac{k}{2} \rfloor}(\bar{\mathbf{g}})$. (Note that we allow for the possibility that $l \in \mathcal{D}_{\lfloor \frac{k}{2} \rfloor}(\bar{\mathbf{g}})$). Therefore, a strictly higher number of links is deleted in E'_j than in E_j. Since $E'_j \subset D(\overline{\mathbf{g}})$, we know from the above argument that a strictly higher number of links is deleted when eliminating E_i than E_j. Assume next that there are is a set of at least two agents, denoted by $\tilde{I}(\overline{\mathbf{g}})$, such that $\tilde{I}(\overline{\mathbf{g}}) \subseteq E_j$ and $\tilde{I}(\overline{\mathbf{g}}) \subseteq I(\overline{\mathbf{g}})$. Then the set E'_j, given by $E'_j = \{E_j \setminus \tilde{I}(\overline{\mathbf{g}})\} \cup \tilde{D}(\overline{\mathbf{g}})$ with $\tilde{D}(\overline{\mathbf{g}}) \subseteq D(\overline{\mathbf{g}})$ and $\tilde{D}(\overline{\mathbf{g}}) \cap E_j = \emptyset$, deletes a strictly higher number of links than E_j. Note that such a set $\tilde{D}(\overline{\mathbf{g}})$ exists, since $e < t(\overline{\mathbf{g}})$. Note first that we can disregard any links between pairs of agents k, l such that in $k \in \tilde{D}(\overline{\mathbf{g}})$ and $l \in \tilde{I}(\overline{\mathbf{g}})$, as these links are deleted in E_j and E'_j. Moreover, any agent $k \in \tilde{D}(\overline{\mathbf{g}})$ is connected to any third agent to which any agent $l \in \tilde{I}(\overline{\mathbf{g}})$ is connected. However, while any pair of agents in $\tilde{I}(\overline{\mathbf{g}})$ is not connected, any pair of agents in $\tilde{D}(\bar{\mathbf{g}})$ is connected. Therefore, a strictly higher number of links is deleted in E'_j than in E_j. Since $E'_j \subset D(\bar{\mathbf{g}})$ we know again by the above argument that a strictly higher number of links is deleted when eliminating E_i than E_j. *Q.E.D.*

Lemma 4: *The vector of Nash equilibrium effort levels,* $\mathbf{x}(\bar{\mathbf{g}}, \gamma)$*, agents' equilibrium payoffs* $v_i(\bar{\mathbf{g}}, \gamma)$ *and deviation effort levels,* $\bar{x}'_i(\bar{\mathbf{g}}+\bar{g}_{i,j}, \gamma)$ *and* $\bar{x}'_i(\bar{\mathbf{g}} - \sum_{j\in D_i(\mathbf{g}'_i,\bar{\mathbf{g}})} \bar{g}_{i,j}, \gamma)$ *and corresponding deviation payoffs* $\Delta v_i(\bar{\mathbf{g}} + \bar{g}_{i,j}, \gamma)$ *and* $\Delta v_i(\bar{\mathbf{g}} - \sum_{j\in D_i(\mathbf{g}'_i,\bar{\mathbf{g}})} \bar{g}_{i,j}, \gamma)$ *are continuous in* γ*.*

Proof. From Ballester et al. (2006) we know that Nash equilibrium effort levels can be written as $x_i(\bar{\mathbf{g}}, \gamma) = \alpha b_i(\bar{\mathbf{g}}, \lambda/\beta)/\left(\beta + \gamma b(\bar{\mathbf{g}}, \lambda/\beta)\right)$.[34] That is, given our assumptions on parameters, Nash equilibrium effort levels are continuous in γ. For a given network $\bar{\mathbf{g}}$, we can then write the relevant expressions in terms of the characterization of Ballester et al. (2006) as functions of γ. In the following we show the case for $v_i(\bar{\mathbf{g}}, \gamma)$, but the same arguments can be used to show that $\bar{x}_i(\bar{\mathbf{g}} - \sum_{j\in D_i(\mathbf{g}'_i,\bar{\mathbf{g}})} \bar{g}_{i,j}, \gamma)$ is continuous as well. For each agent j write $x_j(\gamma) : [0, \lambda] \to \mathbb{R}$ with $x_j(\bar{\mathbf{g}}, \gamma) = \alpha b_j(\bar{\mathbf{g}}, \lambda/\beta)/\left(\beta + \gamma b(\bar{\mathbf{g}}, \lambda/\beta)\right)$, define $f(\gamma) :\ [0, \lambda] \to \mathbb{R}$ with $f(\gamma) = \gamma$. Since the product of two continuous functions is continuous, we know that $\gamma x_j(\bar{\mathbf{g}}, \gamma)$ is continuous in γ. Since the sum of two continuous functions is continuous, we know that $\alpha + \lambda \sum_{j\in N_i(\bar{\mathbf{g}})} x_j(\gamma) - \gamma \sum_{j\neq i} x_j(\gamma)$ is continuous. Define $h(\gamma) :\ [0, \lambda] \to \mathbb{R}$ as $h(\gamma) = \alpha + \lambda \sum_{j\in N_i(\bar{\mathbf{g}})} x_j(\gamma) - \gamma \sum_{j\neq i} x_j(\gamma)$. Finally, define $g(x) :\ \mathbb{R} \to \mathbb{R}$ with $g(x) = x^2$. We can then write $v_i(\bar{\mathbf{g}}, \gamma) = (g \circ h)(\gamma)$. Since the composition of two continuous functions is continuous, $v_i(\bar{\mathbf{g}}, \gamma)$ is continuous in γ. The expression for $\bar{x}'_i(\bar{\mathbf{g}}+\bar{g}_{i,j}, \gamma)$ is given by

$$\bar{x}'_i(\bar{\mathbf{g}}+\bar{g}_{i,j}, \gamma) = a\left(b - cz_i(\bar{\mathbf{g}}, \gamma) + d(\gamma z_j(\bar{\mathbf{g}}, \gamma) - \lambda y_j(\bar{\mathbf{g}}, \gamma)) + ey_i(\bar{\mathbf{g}}, \gamma)\right),$$

where $a = 1/\left((\beta + \gamma)(\beta + 2\gamma - \lambda)(\beta + \lambda)\right)$, $b = \alpha(\beta + 2\gamma)(\beta + \gamma)$, $c = \gamma(\beta^2 + 2\beta + \gamma\lambda)$, $d = -\lambda(\beta + \gamma)$, $e = (\beta^2 + 2\beta + \lambda\gamma)$. Since we can again write $z_i(\bar{\mathbf{g}}, \gamma)$, $z_j(\bar{\mathbf{g}}, \gamma)$, $y_i(\bar{\mathbf{g}}, \gamma)$ and $y_j(\bar{\mathbf{g}}, \gamma)$ in terms of the Bonacich centralities, it follows by the above arguments that $\bar{x}'_i(\bar{\mathbf{g}}+\bar{g}_{i,j}, \gamma)$ is continuous in γ. We can then use the same arguments as above to show that $\Delta v_i(\bar{\mathbf{g}} + \bar{g}_{i,j}, \gamma)$ is continuous in γ. *Q.E.D.*

[34] Ballester et al. (2006) characterize the Nash equilibrium effort levels for fixed $\bar{\mathbf{g}}$ in terms of agents' Bonacich centralities as follows. Denote by $\bar{g}^{[k]}_{i,j} \geq 0$ the number of paths between i and j of length k in $\bar{\mathbf{g}}$. Let $m_{i,j}(\bar{\mathbf{g}}, \frac{\lambda}{\beta}) = \sum_{k=0}^{\infty} (\frac{\lambda}{\beta})^k \bar{g}^{[k]}_{i,j}$. That is $m_{i,j}(\bar{\mathbf{g}}, \frac{\lambda}{\beta})$ counts the number of paths that start at i and end at j and paths of length k are weighted by $(\frac{\lambda}{\beta})^k$. The Bonacich centrality of a node i is then given by $b_i(\bar{\mathbf{g}}, \frac{\lambda}{\beta}) = \sum_{j=1}^{n} m_{i,j}(\bar{\mathbf{g}}, \frac{\lambda}{\beta})$. Note that $b_i(\bar{\mathbf{g}}, \frac{\lambda}{\beta}) = m_{i,i}(\bar{\mathbf{g}}, \frac{\lambda}{\beta}) + \sum_{j\neq i} m_{i,j}(\bar{\mathbf{g}}, \frac{\lambda}{\beta})$. By definition $m_{i,i}(\bar{\mathbf{g}}, \frac{\lambda}{\beta}) \geq 1$ and therefore $b_i(\bar{\mathbf{g}}, \frac{\lambda}{\beta}) \geq 1$. Denote the sum of Bonacich centralities by $b(\bar{\mathbf{g}}, \frac{\lambda}{\beta}) = \sum_{j=1}^{n} b_j(\bar{\mathbf{g}}, \frac{\lambda}{\beta})$. Given our assumptions on parameters, a Nash equilibrium on a fixed network $\bar{\mathbf{g}}$ always exists and the unique Nash equilibrium effort level of an agent i is given by $x_i(\bar{\mathbf{g}}) = \alpha b_i(\bar{\mathbf{g}}, \frac{\lambda}{\beta})/\left(\beta + \gamma b(\bar{\mathbf{g}}, \frac{\lambda}{\beta})\right)$ (Ballester et al. (2006), Proposition 1).

Lemma 5: *If $\gamma = 0$, then $x_k(\bar{\mathbf{g}}, 0) - x_k(\bar{\mathbf{g}}^{-E}, 0) > 0$ for all $k \in N \setminus E$ such that there exists an agent $i \in E$ and $k \in N^i(\bar{\mathbf{g}})$, while $x_k(\bar{\mathbf{g}}, 0) - x_k(\bar{\mathbf{g}}^{-E}, 0) = 0$ for all k such that there does not exist an agent $i \in E$ such that $k \in N^i(\bar{\mathbf{g}})$. Furthermore, for γ sufficiently small $x_k(\bar{\mathbf{g}}, \gamma) - x_k(\bar{\mathbf{g}}^{-E}, \gamma)$ is arbitrarily close to $x_k(\bar{\mathbf{g}}, 0) - x_k(\bar{\mathbf{g}}^{-E}, 0)$ for all $k \in N \setminus E$.*

Proof. Assume first that $\gamma = 0$. Note that then $\bar{x}_i(0, z_i) = \bar{x}_i(0, 0)$ $\forall z_i$ and effort levels are bounded below by $\bar{x}_i(0, 0) = \bar{x}(0, 0)$. Moreover, $\partial \bar{x}(y, z)/\partial y > 0$, while $\partial \bar{x}(y, z)/\partial z = 0$ $\forall y, z$. Note next that, since $\gamma = 0$, we can treat different components in $\bar{\mathbf{g}}$ in isolation. Assume first that there exists an agent $i \in E$ such that $k \in N^i(\bar{\mathbf{g}})$. Then for any agent l such that $l \in N^i(\bar{\mathbf{g}})$ and $\bar{g}_{l,m} = 1$ for some $m \in E$, we have that $y_l(\bar{\mathbf{g}}^{-E}, 0) = \sum_{j \in N_l(\bar{\mathbf{g}}^{-E})} x_j(\bar{\mathbf{g}}, 0) < \sum_{j \in N_l(\bar{\mathbf{g}})} x_j(\bar{\mathbf{g}}, 0) = y_l(\bar{\mathbf{g}}, 0)$. Iterating on best responses, effort levels of each agent $k \in N^i(\bar{\mathbf{g}})$ in $\bar{\mathbf{g}}^{-E}$ are a weakly decreasing sequence of real numbers numbers, such that each agent strictly decreases her effort level in some iteration. Since the sequence is bounded below by $\bar{x}(0, 0)$, effort levels converge to the unique Nash equilibrium effort levels in $\bar{\mathbf{g}}^{-E}$ with $x_k(\bar{\mathbf{g}}^{-E}, 0) < x_k(\bar{\mathbf{g}}, 0)$ for all $k \in N^i(\bar{\mathbf{g}})$. Note next that for any agent k such that there does not exist an agent $i \in E$ such that $k \in N^i(\bar{\mathbf{g}})$, $\sum_{j \in N_k(\bar{\mathbf{g}}^{-E})} x_j(\bar{\mathbf{g}}, 0) = \sum_{j \in N_k(\bar{\mathbf{g}})} x_j(\bar{\mathbf{g}}, 0)$ holds and no agent has an incentive to adjust effort levels. That is, $x_k(\bar{\mathbf{g}}^{-E}, 0) = x_k(\bar{\mathbf{g}}, 0)$ for all agents k such that there does not exist an agent $i \in E$ such that $k \in N^i(\bar{\mathbf{g}})$. Assume next that γ is sufficiently small. From Lemma 4 we know that the vector of Nash equilibrium effort levels, $\mathbf{x}(\bar{\mathbf{g}}, \gamma)$, changes continuously in γ for any $\bar{\mathbf{g}}$. The second statement then follows directly from $lim_{\gamma \to 0} x_k(\bar{\mathbf{g}}, \gamma) = x_k(\bar{\mathbf{g}}, 0)$ and $lim_{\gamma \to 0} x_k(\bar{\mathbf{g}}^{-E}, \gamma) = x_k(\bar{\mathbf{g}}^{-E}, 0)$. *Q.E.D.*

Lemma 6: *If $\gamma = 0$ and $\hat{\bar{\mathbf{g}}} \subset \bar{\mathbf{g}}$, then for any agent k such that there exists an agent $i \in N^k(\bar{\mathbf{g}})$* with *$\bar{g}_{i,j} = 1$ and $\hat{\bar{g}}_{i,j} = 0$, $x_k(\bar{\mathbf{g}}, 0) - x_k(\hat{\bar{\mathbf{g}}}, 0) > 0$ holds, while for any agent k such that there does not exist an agent $i \in N^k(\bar{\mathbf{g}})$* with *$\bar{g}_{i,j} = 1$ and $\hat{\bar{g}}_{i,j} = 0$, $x_k(\bar{\mathbf{g}}, 0) - x_k(\hat{\bar{\mathbf{g}}}, 0) = 0$ holds. Furthermore, for γ sufficiently small $x_k(\bar{\mathbf{g}}, \gamma) - x_k(\hat{\bar{\mathbf{g}}}, \gamma)$ is arbitrarily close to $x_k(\bar{\mathbf{g}}, 0) - x_k(\hat{\bar{\mathbf{g}}}, 0)$ for any agent $k \in N$.*

Proof. Assume first that $\gamma = 0$. Note that then again $\bar{x}_i(0, z_i) = \bar{x}_i(0, 0)$ $\forall z_i$ and effort levels are bounded below by $\bar{x}_i(0, 0) = \bar{x}(0, 0)$. Moreover, $\partial \bar{x}(y, z)/\partial y > 0$, while $\partial \bar{x}(y, z)/\partial z = 0$ $\forall y, z$. Note also that, since $\hat{\bar{\mathbf{g}}} \subset \bar{\mathbf{g}}$, $\hat{\bar{\mathbf{g}}}$ can be obtained from $\bar{\mathbf{g}}$ by deleting any links such that $\bar{g}_{i,j} = 1$ and $\hat{\bar{g}}_{i,j} = 0$. Moreover, since $\gamma = 0$ and $\hat{\bar{\mathbf{g}}} \subset \bar{\mathbf{g}}$, we can again analyze components in $\bar{\mathbf{g}}$ in isolation. If there does not exist an agent $i \in N^k(\bar{\mathbf{g}})$ with $\bar{g}_{i,j} = 1$ while $\hat{\bar{g}}_{i,j} = 0$, then $N^k(\bar{\mathbf{g}}) = N^k(\hat{\bar{\mathbf{g}}})$ and $x_k(\hat{\bar{\mathbf{g}}}, 0) = x_k(\bar{\mathbf{g}}, 0)$. To analyze effort levels of agents k such that there exists an agent $i \in N^k(\bar{\mathbf{g}})$ with $\bar{g}_{i,j} = 1$ and $\hat{\bar{g}}_{i,j} = 0$, we consider agents' best responses to the Nash equilibrium effort levels for $\bar{\mathbf{g}}$, $\mathbf{x}(\bar{\mathbf{g}}, 0)$, when the network is $\hat{\bar{\mathbf{g}}}$. There then exists an agent $i \in N^k(\hat{\bar{\mathbf{g}}})$ (where we allow for $i = k$) such that $\sum_{j \in N_i(\hat{\bar{\mathbf{g}}})} x_j(\bar{\mathbf{g}}, 0) < \sum_{j \in N_i(\bar{\mathbf{g}})} x_j(\bar{\mathbf{g}}, 0)$. Again iterating on best responses, effort levels of each agent $l \in N^k(\hat{\bar{\mathbf{g}}})$ in $\hat{\bar{\mathbf{g}}}$ are a weakly decreasing sequence of real numbers numbers, where each agent strictly decreases her effort level in some iteration. Since the sequence is bounded below by $\bar{x}(0,0)$, effort levels converge to the unique Nash equilibrium effort levels for $\hat{\bar{\mathbf{g}}}$ with $x_k(\hat{\bar{\mathbf{g}}},0) < x_k(\bar{\mathbf{g}},0)$ for all $l \in N^k(\hat{\bar{\mathbf{g}}})$ and therefore $x_k(\hat{\bar{\mathbf{g}}},0) < x_k(\bar{\mathbf{g}},0)$. Assume next that γ is sufficiently small. From Lemma 4 we know that the vector of Nash equilibrium effort levels, $\mathbf{x}(\hat{\bar{g}},\gamma)$, changes continuously in γ for any $\hat{\bar{g}}$, so that the second statement follows directly from $lim_{\gamma\to 0}x_k(\hat{\bar{\mathbf{g}}},\boldsymbol{\gamma}) = x_k(\hat{\bar{\mathbf{g}}},0)$ and $lim_{\gamma\to 0}x_k(\bar{\mathbf{g}},\boldsymbol{\gamma}) = x_k(\bar{\mathbf{g}},0)$. *Q.E.D.*

Lemma 7: *Assume agents play their Nash equilibrium effort levels, $\mathbf{x}(\bar{\mathbf{g}})$, in network $\bar{\mathbf{g}}$. Then a minimal deletion best response, $\mathbf{g}_i'^m$, always exists and is unique for every agent $i \in N$. Furthermore, if $\mathbf{g}_i'$ is such that $k \in N_i(\bar{\mathbf{g}}')$ and $l \in D_i(\mathbf{g}_i',\bar{\mathbf{g}})$, then $x_k(\bar{\mathbf{g}}) > x_l(\bar{\mathbf{g}})$. Moreover, if $\mathbf{g}_i'^m$ and another deletion best response $\mathbf{g}_i'$ are such that $\mid N_i(\bar{\mathbf{g}}'^m) \mid=\mid N_i(\bar{\mathbf{g}}') \mid$, then $N_i(\bar{\mathbf{g}}'^m) = N_i(\bar{\mathbf{g}}')$, while if $\mid N_i(\bar{\mathbf{g}}'^m) \mid<\mid N_i(\bar{\mathbf{g}}') \mid$ then $N_i(\bar{\mathbf{g}}'^m) \subset N_i(\bar{\mathbf{g}}')$.*

Proof. Recall that $z_i(\bar{\mathbf{g}}) = \sum_{j\in N\setminus\{i\}} x_j$, so that in any deviation by agent i, $z_i(\bar{\mathbf{g}}) = z_i(\bar{\mathbf{g}}')$ holds. We can therefore treat $v(y_i,z_i)$ as a strictly convex function in y_i when considering deletion strategies. From $\mathbf{g}_i'^m \subseteq \mathbf{g}_i$ we know that $N_i(\bar{\mathbf{g}}') \subseteq N_i(\bar{\mathbf{g}})$ holds. We next show in two steps that any deletion best response $\mathbf{g}_i' \in argmax_{\mathbf{g}_i'\colon \mathbf{g}_i'\subseteq\mathbf{g}_i}\Pi_i(\mathbf{g}_i',\bar{\mathbf{g}})$ is such that for all j and k with $j \in N_i(\bar{\mathbf{g}}')$ and $k \in D_i(\mathbf{g}_i',\bar{\mathbf{g}})$, $x_j(\bar{\mathbf{g}}) > x_k(\bar{\mathbf{g}})$ holds. Assume that $\mathbf{g}_i'$ is a deletion best response, such that there exists a pair of agents j and k with $x_j(\bar{\mathbf{g}}) > x_k(\bar{\mathbf{g}})$ and $j \in D_i(\mathbf{g}_i',\bar{\mathbf{g}})$ and $k \in N_i(\bar{\mathbf{g}}')$. But then deletion strategy $\mathbf{g}_i''= \mathbf{g}_i' + g_{i,j} - g_{i,k}$ yields strictly higher deviation payoffs, since $y_i(\bar{\mathbf{g}}'') > y_i(\bar{\mathbf{g}}')$, while $z_i(\bar{\mathbf{g}}'') = z_i(\bar{\mathbf{g}}')$ and $\partial v(y_i,z_i)/\partial y_i > 0$. Next we show that for any deletion best response, if $j \in D_i(\mathbf{g}_i',\bar{\mathbf{g}})$, then for any $k \in N_i(\bar{\mathbf{g}})$ such that $x_k(\bar{\mathbf{g}}) = x_j(\bar{\mathbf{g}})$, $k \in D_i(\mathbf{g}_i',\bar{\mathbf{g}})$ also holds. Assume that $\mathbf{g}_i'$ is a deletion best response and, contrary to the above, that $j \in D_i(\mathbf{g}_i',\bar{\mathbf{g}})$, but $k \notin D_i(\mathbf{g}_i',\bar{\mathbf{g}})$ and $k \in N_i(\bar{\mathbf{g}})$ with $x_k(\bar{\mathbf{g}}) = x_j(\bar{\mathbf{g}})$. Since $\mathbf{g}_i'$ is a deletion best response and $j \in D_i(\mathbf{g}_i',\bar{\mathbf{g}})$, it must be the case that $\kappa \geq v(y_i(\bar{\mathbf{g}}') + x_j(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})) - v(y_i(\bar{\mathbf{g}}'), z_i(\bar{\mathbf{g}})$ holds. However, since v is strictly convex in y_i and $x_k(\bar{\mathbf{g}}) = x_j(\bar{\mathbf{g}})$, $v(y_i(\bar{\mathbf{g}}') + x_j(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})) - v(y_i(\bar{\mathbf{g}}'), z_i(\bar{\mathbf{g}}) > v(y_i(\bar{\mathbf{g}}'), z_i(\bar{\mathbf{g}})) - v(y_i(\bar{\mathbf{g}}') - x_k(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})$ also holds and therefore $\kappa > v(y_i(\bar{\mathbf{g}}'), z_i(\bar{\mathbf{g}})) - v(y_i(\bar{\mathbf{g}}') - x_k(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})$ holds. That is, deviation payoffs are strictly larger in the deviation $\mathbf{g}_i''= \mathbf{g}_i' - g_{i,k}$ than in $\mathbf{g}_i'$, and $\mathbf{g}_i'$ is not a deletion best response. Therefore, in any deletion best response $\mathbf{g}_i' \in argmax_{\mathbf{g}_i'\colon \mathbf{g}_i'\subseteq\mathbf{g}_i}\Pi_i(\mathbf{g}_i',\bar{\mathbf{g}})$, if $k \in D_i(\mathbf{g}_i',\bar{\mathbf{g}})$ and $j \in N_i(\bar{\mathbf{g}}')$, then $x_j(\bar{\mathbf{g}}) > x_k(\bar{\mathbf{g}})$. Note that then in any deletion best response $\mathbf{g}_i'$ with agents j and k such that $j,k \in N_i(\bar{\mathbf{g}})$ and $x_k(\bar{\mathbf{g}}) = x_j(\bar{\mathbf{g}})$, either $j,k \in D_i(\mathbf{g}_i',\bar{\mathbf{g}})$ or $j,k \notin D_i(\mathbf{g}_i',\bar{\mathbf{g}})$. The above then allows us to characterize minimal deletion best responses as follows. Partition the set of agents in $N_i(\bar{\mathbf{g}})$ by their effort levels in $\bar{\mathbf{g}}$. Assume that there are m distinct effort levels in $N_i(\bar{\mathbf{g}})$. Denote by $N_i^1(\bar{\mathbf{g}})$ the set of agents with the lowest effort levels in $N_i(\bar{\mathbf{g}})$, $N_i^2(\bar{\mathbf{g}})$ is the set of agents with the second lowest effort levels in $N_i(\bar{\mathbf{g}})$, and so forth, until $N_i^m(\bar{\mathbf{g}})$, the set of agent with the highest effort levels in $N_i(\bar{\mathbf{g}})$. From the above we know that any deletion best response, $\mathbf{g}_i'$, is such that either $D_i(\mathbf{g}_i',\bar{\mathbf{g}}) = \emptyset$, or $D_i(\mathbf{g}_i',\bar{\mathbf{g}}) = \cup_{j=1}^k N_i^j(\bar{\mathbf{g}})$ for some integer k with $1 \leq k \leq m$. Finally, the unique $\mathbf{g}_i'^m$ is then such that k is maximal, $\mathbf{g}_{i,j}'^m = 0\ \forall j \notin N_i(\bar{\mathbf{g}}')$. Note that we then also know that, if $k \in N_i(\bar{\mathbf{g}}'^m)$ and $l \in D_i(\mathbf{g}_i'^m, \bar{\mathbf{g}})$, then $x_k(\bar{\mathbf{g}}) > x_l(\bar{\mathbf{g}})$. Finally, it follows directly from the above that, if $\mathbf{g}_i'$ and $\mathbf{g}_i'^m$ are such that $\mid N_i(\bar{\mathbf{g}}'^m) \mid = \mid N_i(\bar{\mathbf{g}}') \mid$, then $N_i(\bar{\mathbf{g}}'^m) = N_i(\bar{\mathbf{g}}')$, while if $\mid N_i(\bar{\mathbf{g}}'^m) \mid < \mid N_i(\bar{\mathbf{g}}') \mid$, then $N_i(\bar{\mathbf{g}}'^m) \subset N_i(\bar{\mathbf{g}}')$. *Q.E.D.*

Lemma 8: *If γ is sufficiently small, $\hat{\bar{\mathbf{g}}} \subseteq \bar{\mathbf{g}}$ and agents play their Nash equilibrium effort levels, $\mathbf{x}(\hat{\bar{\mathbf{g}}})$ and $\mathbf{x}(\bar{\mathbf{g}})$, then an agent i's minimal deletion best response in $\hat{\bar{\mathbf{g}}}$, $\hat{\mathbf{g}}_i'^m$, and agent i's minimal deletion best response in $\bar{\mathbf{g}}$, $\mathbf{g}_i'^m$, are such that $N_i(\hat{\bar{\mathbf{g}}}') \subseteq N_i(\bar{\mathbf{g}}')$.*

Proof. Assume first that $\hat{\bar{\mathbf{g}}} = \bar{\mathbf{g}}$ and γ is sufficiently small. By Lemma 7 the minimal deletion best response is unique and therefore $N_i(\hat{\bar{\mathbf{g}}}') = N_i(\bar{\mathbf{g}}')$. Assume next that $\hat{\bar{\mathbf{g}}} \subset \bar{\mathbf{g}}$ holds and that γ is sufficiently small. Note first that since $\hat{\bar{\mathbf{g}}} \subset \bar{\mathbf{g}}$, $\eta_i(\bar{\mathbf{g}}) \geq \eta_i(\hat{\bar{\mathbf{g}}})$ holds $\forall i \in N$. Note next that the above statement holds trivially if either $\eta_i(\bar{\mathbf{g}}) = 0$, $\eta_i(\hat{\bar{\mathbf{g}}}) = 0$, or $\eta_i(\hat{\bar{\mathbf{g}}}') = 0$. We therefore assume that $\eta_i(\bar{\mathbf{g}}) \geq 1$, $\eta_i(\hat{\bar{\mathbf{g}}}) \geq 1$ and $\eta_i(\hat{\bar{\mathbf{g}}}') \geq 1$. Denote by $\hat{\bar{\mathbf{g}}}'^m$ the network that is obtained from agent i's minimal deletion best response, $\hat{\mathbf{g}}_i'^m$, in $\hat{\bar{\mathbf{g}}}$, while $\bar{\mathbf{g}}'^m$ is obtained from agent i's minimal deletion best response, $\mathbf{g}_i'^m$, in $\bar{\mathbf{g}}$. Pick a ranking of agents in the set of agent i's neighbors *after* proposed deviation in $\hat{\bar{\mathbf{g}}}$, $N_i(\hat{\bar{\mathbf{g}}}'^m)$, such that $x_1(\hat{\bar{\mathbf{g}}}'^m) \geq x_2(\hat{\bar{\mathbf{g}}}'^m) \geq ... \geq x_{\eta_i(\hat{\bar{\mathbf{g}}}'^m)}(\hat{\bar{\mathbf{g}}}'^m)$, where we use the subscript to refer to the position in the ranking rather than to an agent's label in the set N. Denote this ranking by $r(N_i(\hat{\bar{\mathbf{g}}}'^m))$. Similarly, pick a ranking of agents in the set of agent i's neighbors *prior* to proposed deviation in $\bar{\mathbf{g}}$, $N_i(\bar{\mathbf{g}})$, such that $x_1(\bar{\mathbf{g}}) \geq x_2(\bar{\mathbf{g}}) \geq ... \geq x_{\eta_i(\bar{\mathbf{g}})}(\bar{\mathbf{g}})$. Denote this ranking with $r(N_i(\bar{\mathbf{g}}))$. Note that since $\hat{\bar{\mathbf{g}}} \subset \bar{\mathbf{g}}$ holds and $\mathbf{g}'^m$ is a deletion best response, $N_i(\hat{\bar{\mathbf{g}}}'^m) \subseteq N_i(\bar{\mathbf{g}})$ holds. Pick an agent $j \in N_i(\hat{\bar{\mathbf{g}}}'^m)$ such that $x_k(\bar{\mathbf{g}}) \geq x_j(\bar{\mathbf{g}})$ $\forall k \in N_i(\hat{\bar{\mathbf{g}}}'^m)$, i.e. pick an agent j in $N_i(\hat{\bar{\mathbf{g}}}'^m)$, such that j's effort level in the network $\bar{\mathbf{g}}$ is weakly smaller than the effort level of any other agent in $N_i(\hat{\bar{\mathbf{g}}}'^m)$ in the network $\bar{\mathbf{g}}$. Out of all agents in the ranking $r(N_i(\bar{\mathbf{g}}))$ with effort level equal to $x_j(\bar{\mathbf{g}})$, pick the agent with the largest subscript in $r(N_i(\bar{\mathbf{g}}))$ and denote the corresponding subscript by t. Defined a truncated ranking of $r(N_i(\bar{\mathbf{g}}))$, denoted by $r_t(N_i(\bar{\mathbf{g}}))$, such that $x_1(\bar{\mathbf{g}}) \geq x_2(\bar{\mathbf{g}}) \geq ... \geq x_{t-1}(\bar{\mathbf{g}}) \geq x_t(\bar{\mathbf{g}})$. Note that, since $x_j(\bar{\mathbf{g}})$ was chosen such that all agents in $N_i(\hat{\bar{\mathbf{g}}}'^m)$ display weakly higher effort levels in $\bar{\mathbf{g}}$, and since t is the agent with the highest subscript in $r(N_i(\bar{\mathbf{g}}))$ such that $x_t(\bar{\mathbf{g}}) = x_j(\bar{\mathbf{g}})$, all agents in $N_i(\hat{\bar{\mathbf{g}}}'^m)$ are included in the ranking $r_t(N_i(\bar{\mathbf{g}}))$. We next show that $x_t(\bar{\mathbf{g}}) > x_{\eta_i(\hat{\bar{\mathbf{g}}}'^m)}(\hat{\bar{\mathbf{g}}}'^m)$. Assume first that $j \in N_i(\hat{\bar{\mathbf{g}}}'^m)$ is an agent with the (weakly) lowest effort level in $r(N_i(\hat{\bar{\mathbf{g}}}'^m))$, i.e. $x_j(\hat{\bar{\mathbf{g}}}'^m) = x_{\eta_i(\bar{\mathbf{g}})}(\bar{\mathbf{g}})$. Since $\hat{\bar{\mathbf{g}}} \subseteq \bar{\mathbf{g}}$ and j is in the same component as i in $\bar{\mathbf{g}}$, we know from Lemma 6 that effort levels for each agent are strictly lower in $\hat{\bar{\mathbf{g}}}$ than in $\bar{\mathbf{g}}$ for γ sufficiently small. Therefore $x_j(\bar{\mathbf{g}}) > x_j(\hat{\bar{\mathbf{g}}}'^m) = x_{\eta_i(\bar{\mathbf{g}})}(\bar{\mathbf{g}})$. Assume next that $j \in N_i(\hat{\bar{\mathbf{g}}}'^m)$ is not an agent with the (weakly) lowest effort level in $r(N_i(\hat{\bar{\mathbf{g}}}'^m))$, i.e. $x_j(\hat{\bar{\mathbf{g}}}'^m) > x_{\eta_i(\bar{\mathbf{g}})}(\bar{\mathbf{g}})$. But then by the previous argument $x_j(\bar{\mathbf{g}}) > x_j(\hat{\bar{\mathbf{g}}}'^m) > x_{\eta_i(\bar{\mathbf{g}})}(\bar{\mathbf{g}})$. Note next that, since $\hat{\mathbf{g}}_i'^m$ is agent i's minimal deletion best response in $\hat{\bar{\mathbf{g}}}$, agent i does not find it profitable to delete any further links in $N_i(\hat{\bar{\mathbf{g}}}'^m)$. From Lemma 7 we know that we only need to consider deviations $\hat{\mathbf{g}}_i'$ such that $x_k(\hat{\bar{\mathbf{g}}}) > x_l(\hat{\bar{\mathbf{g}}})$ holds $\forall k, l : k \in N_i(\hat{\bar{\mathbf{g}}})$ and $l \in D_i(\hat{\mathbf{g}}_i', \hat{\bar{\mathbf{g}}})$ and we can therefore summarize the conditions, such that agent i does not find it profitable to delete any links in $N_i(\hat{\bar{\mathbf{g}}}'^m)$ as follows:

$$(A)\quad \frac{v(\sum_{j=1}^{\eta_i(\hat{\bar{\mathbf{g}}}'^m)} x_j(\hat{\bar{\mathbf{g}}}'^m), z_i(\hat{\bar{\mathbf{g}}}'^m)) - v(\sum_{j=1}^{\eta_i(\hat{\bar{\mathbf{g}}}'^m)} x_j(\hat{\bar{\mathbf{g}}}'^m) - \sum_{j=0}^{k} x_{\eta_i(\hat{\bar{\mathbf{g}}}'^m)-j}(\hat{\bar{\mathbf{g}}}'^m), z_i(\hat{\bar{\mathbf{g}}}'^m))}{k+1}$$

$$> \kappa$$

for all $k \in \mathbb{N} : 0 \leq k \leq \eta_i(\hat{\bar{\mathbf{g}}}'^m) - 1$. To see this, note first that if

$$\kappa > \frac{v(\sum_{j=1}^{\eta_i(\hat{\bar{\mathbf{g}}}'^m)} x_j(\hat{\bar{\mathbf{g}}}'^m), z_i(\hat{\bar{\mathbf{g}}}'^m)) - v(\sum_{j=1}^{\eta_i(\hat{\bar{\mathbf{g}}}'^m)} x_j(\hat{\bar{\mathbf{g}}}'^m) - \sum_{j=0}^{k} x_{\eta_i(\hat{\bar{\mathbf{g}}}'^m)-j}(\hat{\bar{\mathbf{g}}}'^m), z_i(\hat{\bar{\mathbf{g}}}'^m))}{k+1}$$

for some $k \in \mathbb{N} : 0 \leq k \leq \eta_i(\hat{\bar{\mathbf{g}}}'^m) - 1$, then agent i can increase deviation payoffs by deleting links to some subset of agents in $N_i(\hat{\bar{\mathbf{g}}}'^m)$. Therefore, $\hat{\mathbf{g}}_i'^m$ is not optimal and hence not a minimum deletion best response. If

$$\kappa = \frac{v(\sum_{j=1}^{\eta_i(\hat{\bar{\mathbf{g}}}'^m)} x_j(\hat{\bar{\mathbf{g}}}'^m), z_i(\hat{\bar{\mathbf{g}}}'^m)) - v(\sum_{j=1}^{\eta_i(\hat{\bar{\mathbf{g}}}'^m)} x_j(\hat{\bar{\mathbf{g}}}'^m) - \sum_{j=0}^{k} x_{\eta_i(\hat{\bar{\mathbf{g}}}'^m)-j}(\hat{\bar{\mathbf{g}}}'^m), z_i(\hat{\bar{\mathbf{g}}}'^m))}{k+1}$$

for some $k \in \mathbb{N} : 0 \leq k \leq \eta_i(\hat{\bar{\mathbf{g}}}'^m) - 1$, then there exists a deviation, $\tilde{\mathbf{g}}_i'$, that yields the same deviation payoffs as $\hat{\mathbf{g}}_i'^m$, but $\tilde{\mathbf{g}}_i' \subset \hat{\mathbf{g}}_i'^m$ holds and $\hat{\mathbf{g}}_i'^m$ is therefore not a minimal deletion best response. Next we show that the conditions in (A) imply that a deviation by agent i in $\bar{\mathbf{g}}$, $\mathbf{g}_i'$, such that agent i keeps his links with the first $\eta_i(\hat{\bar{\mathbf{g}}}'^m)$ agents in the ranking $r(N_i(\bar{\mathbf{g}}))$, but deletes all other links, yields strictly higher payoffs than a deviation where any further links in $N_i(\bar{\mathbf{g}})$ are deleted. The appropriate conditions are given by

$$(B)\quad \frac{v(\sum_{j=1}^{\eta_i(\hat{\bar{\mathbf{g}}}'^m)} x_j(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})) - v(\sum_{j=1}^{\eta_i(\hat{\bar{\mathbf{g}}}'^m)} x_j(\bar{\mathbf{g}}) - \sum_{j=0}^{k} x_{\eta_i(\hat{\bar{\mathbf{g}}}'^m)-j}(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}}))}{k+1} > \kappa$$

for all $k \in \mathbb{N} : 0 \leq k \leq \eta_i(\hat{\bar{\mathbf{g}}}') - 1$. To see that the conditions in (A) imply the conditions in (B), recall that $N_i(\hat{\bar{\mathbf{g}}}'^m) \subseteq N_i(\bar{\mathbf{g}})$. Note further that, since $\hat{\bar{\mathbf{g}}} \subset \bar{\mathbf{g}}$, $\eta_j(\hat{\bar{\mathbf{g}}}'^m) \geq 1 \; \forall j \in N_i(\hat{\bar{\mathbf{g}}}'^m)$ and, since γ sufficiently small, we know from Lemma 6 that all agents $j \in N_i(\hat{\bar{\mathbf{g}}}'^m)$ display strictly lower effort levels in $\hat{\bar{\mathbf{g}}}$ than in $\bar{\mathbf{g}}$. Since $N_i(\hat{\bar{\mathbf{g}}}'^m) \subseteq N_i(\bar{\mathbf{g}})$ also holds, we know that the first $\eta_i(\hat{\bar{\mathbf{g}}}'^m)$ agents in the ranking $r_t(N_i(\bar{\mathbf{g}}))$ display strictly higher effort levels than agents with the same rank (i.e. the same subscript) as agents in $r(N_i(\hat{\bar{\mathbf{g}}}'^m))$. Note next that $v(y,z)$ is strictly convex in y and, for $\gamma = 0$, $v(y,z)$ can be treated as a function of only the first argument. Since the conditions hold strictly in (A), and since effort levels are strictly larger in (B), we know from the convexity of the value function that they also hold strictly for (B). From Lemma 4 it then follows that they also hold strictly for γ sufficiently small. That is, a deviation strategy, in which all links, except for the first $\eta_i(\hat{\bar{\mathbf{g}}}'^m)$ agents in the ranking $r_t(N_i(\bar{\mathbf{g}}))$ yields strictly higher deviation payoffs than deleting any further agents. Note that if $t = \eta_i(\hat{\bar{\mathbf{g}}}'^m)$, then $N_i(\hat{\bar{\mathbf{g}}}'^m) \subseteq N_i(\bar{\mathbf{g}}'^m)$. Assume next that $t > \eta_i(\hat{\bar{\mathbf{g}}}'^m)$ holds. Consider the condition for $k = 0$ in (A), which reads

$$v(\textstyle\sum_{j=1}^{\eta_i(\hat{\bar{\mathbf{g}}}'^m)} x_j(\hat{\bar{\mathbf{g}}}'^m), z_i(\hat{\bar{\mathbf{g}}}'^m)) - v(\sum_{j=1}^{\eta_i(\hat{\bar{\mathbf{g}}}'^m)} x_j(\hat{\bar{\mathbf{g}}}'^m) - x_{\eta_i(\hat{\bar{\mathbf{g}}}'^m)}(\hat{\bar{\mathbf{g}}}'^m), z_i(\hat{\bar{\mathbf{g}}}'^m)) > \kappa.$$

Note that $\sum_{j=1}^{\eta_i(\hat{\bar{\mathbf{g}}}'^m)} x_j(\bar{\mathbf{g}}^{-E_k}\bar{\mathbf{g}}) > \sum_{j=1}^{\eta_i(\hat{\bar{\mathbf{g}}}'^m)} x_j(\hat{\bar{\mathbf{g}}}'^m)$ and, since $x_t(\bar{\mathbf{g}}) > x_{\eta_i(\hat{\bar{\mathbf{g}}}'^m)}(\hat{\bar{\mathbf{g}}}'^m)$ and $x_t(\hat{\bar{\mathbf{g}}}'^m)$ is the weakly lowest effort level of agents in $r_t(N_i(\bar{\mathbf{g}}))$, we know that $x_j(\bar{\mathbf{g}}) > x_{\eta_i(\hat{\bar{\mathbf{g}}}'^m)}(\hat{\bar{\mathbf{g}}}'^m)$ for all agents j listed in the ranking $r_t(N_i(\bar{\mathbf{g}}))$. We can then use again the conditions in (A) and the convexity of the value function to show that deleting links to agents from agent t to the $\eta_i(\hat{\bar{\mathbf{g}}}'^m)$-th agent in $r_t(N_i(\bar{\mathbf{g}}))$ decreases deviation payoffs. That is

$$\frac{v(\sum_{j=1}^{t} x_j(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})) - v(\sum_{j=1}^{t} x_j(\bar{\mathbf{g}}) - \sum_{j=0}^{k} x_{t-j}(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}}))}{k+1} > \kappa$$

holds for all $k \in \mathbb{N} : 0 \leq k \leq t - \eta_i(\hat{\bar{\mathbf{g}}}'^m) - 1$. Therefore, keeping all links in the ranking $r_t(N_i(\bar{\mathbf{g}}))$ yields strictly higher payoffs than deleting (any subset) of links in $r_t(N_i(\bar{\mathbf{g}}))$ and therefore $N_i(\hat{\bar{\mathbf{g}}}'^m) \subseteq N_i(\bar{\mathbf{g}}'^m)$. *Q.E.D.*

Lemma 9: *Assume γ is sufficiently small, $(\mathbf{x}, \bar{\mathbf{g}})$ is a pairwise Nash equilibrium with $\bar{\mathbf{g}} \notin \{\bar{\mathbf{g}}^e\}$ and $E \in \mathscr{E}(e)$. Then there does not exist a pair of agents $j,k \in N \setminus E$ in any configuration $(\mathbf{x}(\hat{\bar{\mathbf{g}}}^{-E}), \hat{\bar{\mathbf{g}}}^{-E})$ with $\hat{\bar{\mathbf{g}}}^{-E} \subseteq \bar{\mathbf{g}}^{-E}$, such that $\bar{g}_{j,k} = 0$ in $\bar{\mathbf{g}}$ and j and k find it profitable to create a link in configuration $(\mathbf{x}(\hat{\bar{\mathbf{g}}}^{-E}), \hat{\bar{\mathbf{g}}}^{-E})$.*

Proof. Note that, since $\bar{\mathbf{g}}$ is a pairwise Nash equilibrium network, we know from Proposition 1 that $\bar{\mathbf{g}}$ is a nested split graph. Therefore, there is at most one non-trivial component (i.e., at most one component that does not consist of only a singleton agent). Consider first the case such that $\gamma = 0$ and assume that j and k are in the non-trivial component in $\bar{\mathbf{g}}$. Since $E \in \mathscr{E}(e)$ we know that there exists an agent $i \in E$ such that $j, k \in N^i(\bar{\mathbf{g}})$. We show that if $\hat{\bar{g}}_{j,k}^{-E} = 0$, then an agent's marginal payoff of creating the link $\hat{\bar{g}}_{j,k}^{-E\prime} = 1$ is strictly lower in $\hat{\bar{\mathbf{g}}}^{-E}$ than when creating the link $\bar{g}'_{j,k} = 1$ in $\bar{\mathbf{g}}$. Note first that from Lemma 5 and Lemma 6 we know that $x_l(\hat{\bar{\mathbf{g}}}^{-E}, 0) < x_l(\bar{\mathbf{g}}, 0)$ for all agents $l \in N^i(\bar{\mathbf{g}})$ and therefore $y_l(\hat{\bar{\mathbf{g}}}^{-E}, 0) < y_l(\bar{\mathbf{g}}, 0)$ for all agents $l \in N^i(\bar{\mathbf{g}})$. Note further that $x'_j(\hat{\bar{\mathbf{g}}}^{-E} + \bar{g}_{j,k}, 0) < x'_j(\bar{\mathbf{g}} + \bar{g}_{j,k}, 0)$ and $x'_k(\hat{\bar{\mathbf{g}}}^{-E} + \bar{g}_{j,k}, 0) < x'_k(\bar{\mathbf{g}} + \bar{g}_{j,k}, 0)$ holds. To see this, note that when $\gamma = 0$, then we can derive the following expressions,

$$x'_j(\bar{\mathbf{g}} + \bar{g}_{j,k}, 0) - x'_j(\hat{\bar{\mathbf{g}}}^{-E} + \bar{g}_{j,k}, 0) = a\left(\beta(y_j(\bar{\mathbf{g}}, 0) - y_j(\hat{\bar{\mathbf{g}}}^{-E}, 0)) + \lambda(y'_k(\bar{\mathbf{g}}, 0) - y'_k(\hat{\bar{\mathbf{g}}}^{-E}, 0)\right)$$

and

$$x'_k(\bar{\mathbf{g}} + \bar{g}_{j,k}, 0) - x'_k(\hat{\bar{\mathbf{g}}}^{-E} + \bar{g}_{j,k}, 0) = a\left(\beta(y_k(\bar{\mathbf{g}}, 0) - y_k(\hat{\bar{\mathbf{g}}}^{-E}, 0)) + \lambda(y'_j(\bar{\mathbf{g}}, 0) - y'_j(\hat{\bar{\mathbf{g}}}^{-E}, 0)\right),$$

where we set $a = \lambda/(\beta^2 - \lambda^2)$. Note that $a > 0$ since $\beta > (n-1)\lambda$. From $y_j(\hat{\bar{\mathbf{g}}}^{-E}, 0) < y_j(\bar{\mathbf{g}}, 0)$ and $y_k(\hat{\bar{\mathbf{g}}}^{-E}, 0) < y_k(\bar{\mathbf{g}}, 0)$ it then follows that $x'_j(\hat{\bar{\mathbf{g}}}^{-E} + \bar{g}_{j,k}, 0) < x'_j(\bar{\mathbf{g}} + \bar{g}_{j,k}, 0)$ and $x'_k(\hat{\bar{\mathbf{g}}}^{-E} + \bar{g}_{j,k}, 0) <$

$x'_k(\bar{\mathbf{g}}+\bar{g}_{j,k},0)$ also hold. Note next that for $\gamma=0$, we can write the marginal payoffs of agent j and k when creating a link in $\bar{\mathbf{g}}$ and $\hat{\bar{\mathbf{g}}}^{-E}$, denoted by $\Delta v_j(\bar{\mathbf{g}}+\bar{g}_{j,k},0)$, $\Delta v_k(\bar{\mathbf{g}}+\bar{g}_{j,k},0)$, $\Delta v_j(\hat{\bar{\mathbf{g}}}^{-E}+\bar{g}_{j,k},0)$ and $\Delta v_k(\hat{\bar{\mathbf{g}}}^{-E}+\bar{g}_{j,k},0)$, as follows

$$\Delta v_j(\bar{\mathbf{g}}+\bar{g}_{j,k},0)=\tfrac{1}{2(\beta+\gamma)}\left((\alpha+\lambda y_j(\bar{\mathbf{g}},0)+\lambda x'_k(\bar{\mathbf{g}}+\bar{g}_{j,k},0)))^2-(\alpha+\lambda y_j(\bar{\mathbf{g}},0))^2\right),$$

$$\Delta v_k(\bar{\mathbf{g}}+\bar{g}_{j,k},0)=\tfrac{1}{2(\beta+\gamma)}\left(\left(\alpha+\lambda y_k(\bar{\mathbf{g}},0)+\lambda x'_j(\bar{\mathbf{g}}+\bar{g}_{j,k},0))\right)^2-(\alpha+\lambda y_k(\bar{\mathbf{g}},0))^2\right),$$

$$\Delta v_j(\hat{\bar{\mathbf{g}}}^{-E}+\bar{g}_{j,k},0)=\tfrac{1}{2(\beta+\gamma)}\left(\left(\alpha+\lambda y_j(\hat{\bar{\mathbf{g}}}^{-E},0)+\lambda x'_k(\hat{\bar{\mathbf{g}}}^{-E}+\bar{g}_{j,k},0))\right)^2-\left(\alpha+\lambda y_j(\hat{\bar{\mathbf{g}}}^{-E},0)\right)^2\right),$$

$$\Delta v_k(\hat{\bar{\mathbf{g}}}^{-E}+\bar{g}_{j,k},0)=\tfrac{1}{2(\beta+\gamma)}\left(\left(\alpha+\lambda y_k(\hat{\bar{\mathbf{g}}}^{-E},0)+\lambda x'_j(\hat{\bar{\mathbf{g}}}^{-E}+\bar{g}_{j,k},0))\right)^2-\left(\alpha+\lambda y_k(\hat{\bar{\mathbf{g}}}^{-E},0)\right)^2\right).$$

Note that $\Delta v_j(\bar{\mathbf{g}}+\bar{g}_{j,k},0)>\Delta v_j(\hat{\bar{\mathbf{g}}}^{-E}+\bar{g}_{j,k},0)$ since $x'_k(\bar{\mathbf{g}}+\bar{g}_{j,k},0)>x'_k(\hat{\bar{\mathbf{g}}}^{-E}+\bar{g}_{j,k},0)$ and $y_j(\bar{\mathbf{g}},0)>y_j(\hat{\bar{\mathbf{g}}}^{-E},0)$. Likewise, $\Delta v_k(\bar{\mathbf{g}}+\bar{g}_{j,k},0)>\Delta v_k(\hat{\bar{\mathbf{g}}}^{-E}+\bar{g}_{j,k},0)$ since $x'_j(\bar{\mathbf{g}}+\bar{g}_{j,k},0)>x'_j(\hat{\bar{\mathbf{g}}}^{-E}+\bar{g}_{j,k},0)$ and $y_k(\bar{\mathbf{g}},0)>y_k(\hat{\bar{\mathbf{g}}}^{-E},0)$. Since $\bar{\mathbf{g}}$ is a PNE network, either $\kappa\geq\Delta v_j(\bar{\mathbf{g}}+\bar{g}_{j,k},0)$ or $\kappa\geq\Delta v_k(\bar{\mathbf{g}}+\bar{g}_{j,k},0)$ holds. Therefore, $\kappa>\Delta v_j(\hat{\bar{\mathbf{g}}}^{-E}+\bar{g}_{j,k},0)$ or $\kappa>\Delta v_k(\hat{\bar{\mathbf{g}}}^{-E}+\bar{g}_{j,k},0)$ holds for any $\hat{\bar{\mathbf{g}}}^{-E}\subseteq\bar{\mathbf{g}}^{-E}$. Assume next that γ is sufficiently small. From Lemma 4 we know that equilibrium payoffs and deviation payoffs change continuously in γ. Since marginal payoffs from creating a link are strictly smaller in $\hat{\bar{\mathbf{g}}}^{-E}$ than in $\bar{\mathbf{g}}$ for $\gamma=0$, and creating the link is not profitable in $\bar{\mathbf{g}}$ (as $\bar{\mathbf{g}}$ is assumed to be a pairwise Nash equilibrium network), the deviation is also not profitable in $\hat{\bar{\mathbf{g}}}^{-E}$ for γ sufficiently small. Assume next that $\gamma=0$ and that j and k are singletons in $\bar{\mathbf{g}}$, i.e. j and k are not in the non-trivial component in $\bar{\mathbf{g}}$. Recall that $\bar{\mathbf{g}}\notin\{\bar{\mathbf{g}}^e\}$ and therefore there exists an agent m such that $\eta_m(\bar{\mathbf{g}})\geq 1$. We first show that creating the link $\bar{g}'_{j,k}=1$ incurs strictly lower marginal payoffs than creating the link $\bar{g}'_{j,m}=1$. Note that, since $\bar{\mathbf{g}}$ is a PNE network and $\bar{g}_{m,k}=0$, either m does not find it profitable to create a link with k in $\bar{\mathbf{g}}$, or k does not find it profitable to create a link with m in $\bar{\mathbf{g}}$ (or both). Assume first that agent m does not find it profitable to create the link with k in $\bar{\mathbf{g}}$. Next we show that the marginal payoffs to m when linking to k are strictly higher than when agent j links to k in $\bar{\mathbf{g}}$. Since $\eta_m(\bar{\mathbf{g}})\geq 1$ and $\eta_j(\bar{\mathbf{g}})=\eta_k(\bar{\mathbf{g}})=0$, we know from Proposition 1 that $x_m(\bar{\mathbf{g}},0)>x_j(\bar{\mathbf{g}},0)$ and therefore $y_m(\bar{\mathbf{g}},0)>y_j(\bar{\mathbf{g}},0)$. We can derive the following expression, $x'_k(\bar{\mathbf{g}}+\bar{g}_{m,k},0)-x'_k(\bar{\mathbf{g}}+\bar{g}_{j,k},0)=(\lambda^2/(\beta^2-\lambda^2))\,(y_m(\bar{\mathbf{g}},0)-y_j(\bar{\mathbf{g}},0))>0$, where the inequality follows from $\beta>(n-1)\lambda$ and $y_m(\bar{\mathbf{g}},0)>y_j(\bar{\mathbf{g}},0)$. Marginal payoffs of agent m, when linking to agent k in $\bar{\mathbf{g}}$, and of agent j when linking to k, denoted by $\Delta v_m(\bar{\mathbf{g}}+\bar{g}_{m,k},0)$ and $\Delta v_j(\bar{\mathbf{g}}+\bar{g}_{j,k},0)$, respectively, are given by

$$\Delta v_m(\bar{\mathbf{g}}+\bar{g}_{m,k},0)=\tfrac{1}{2(\beta+\gamma)}\left((\alpha+\lambda y_m(\bar{\mathbf{g}},0)+\lambda x'_k(\bar{\mathbf{g}}+\bar{g}_{m,k},0))^2-(\alpha+\lambda y_m(\bar{\mathbf{g}},0))^2\right)$$

and

$$\Delta v_j(\bar{\mathbf{g}}+\bar{g}_{j,k},0)=\tfrac{1}{2(\beta+\gamma)}\left((\alpha+\lambda y_j(\bar{\mathbf{g}},0)+\lambda x'_k(\bar{\mathbf{g}}+\bar{g}_{j,k},0))^2-(\alpha+\lambda y_j(\bar{\mathbf{g}},0))^2\right).$$

Note next that, since $x_m(\bar{\mathbf{g}},0) > x_j(\bar{\mathbf{g}},0)$ (by Proposition 1), we know from the best response functions that $\alpha + \lambda y_m(\bar{\mathbf{g}},0) > \alpha + \lambda y_j(\bar{\mathbf{g}},0)$ holds and since $x'_k(\bar{\mathbf{g}} + \bar{g}_{m,k}, 0) > x'_k(\bar{\mathbf{g}} + \bar{g}_{j,k}, 0)$, $\Delta v_m(\bar{\mathbf{g}} + \bar{g}_{m,k}, 0) > \Delta v_j(\bar{\mathbf{g}} + \bar{g}_{j,k}, 0)$ holds. Since we assumed that m does not find it profitable to create the link with k in $\bar{\mathbf{g}}$, $\kappa \geq \Delta v_m(\bar{\mathbf{g}} + \bar{g}_{m,k}, 0)$ holds and therefore $\kappa > \Delta v_j(\bar{\mathbf{g}} + \bar{g}_{j,k}, 0)$ also holds. Note that, since j and k are singletons and $\gamma = 0$, we know that $\Delta v_j(\bar{\mathbf{g}} + \bar{g}_{j,k}, 0) = \Delta v_j(\hat{\bar{\mathbf{g}}}^{-E} + \bar{g}_{j,k}, 0)$ and therefore $\kappa > \Delta v_j(\hat{\bar{\mathbf{g}}}^{-E} + \bar{g}_{j,k}, 0)$ holds for any $\hat{\bar{\mathbf{g}}}^{-E} \subseteq \bar{\mathbf{g}}^{-E}$. From Lemma 4 it follows that, for γ sufficiently small, $\Delta v_j(\hat{\bar{\mathbf{g}}}^{-E} + \bar{g}_{j,k}, \gamma)$ is arbitrarily close to $\Delta v_j(\bar{\mathbf{g}} + \bar{g}_{j,k}, 0)$ and therefore $\kappa > \Delta v_j(\hat{\bar{\mathbf{g}}}^{-E} + \bar{g}_{j,k}, \gamma)$ holds for any $\hat{\bar{\mathbf{g}}}^{-E} \subseteq \bar{\mathbf{g}}^{-E}$ and any γ sufficiently small. Assume next that agent k does not find it profitable to create the link with m in $\bar{\mathbf{g}}$. One can then use an argument analogous to the one above to show that $x'_m(\bar{\mathbf{g}} + \bar{g}_{m,k}, 0) > x'_j(\bar{\mathbf{g}} + \bar{g}_{j,k}, 0)$ and that $\Delta v_k(\bar{\mathbf{g}} + \bar{g}_{m,k}, 0) > \Delta v_k(\bar{\mathbf{g}} + \bar{g}_{j,k}, 0)$ holds. Since k does not find it profitable to create the link with m, $\kappa \geq \Delta v_k(\bar{\mathbf{g}} + \bar{g}_{m,k}, 0)$ and therefore $\kappa > \Delta v_k(\bar{\mathbf{g}} + \bar{g}_{j,k}, 0)$. Since j and k are singletons and $\gamma = 0$, we know that $\Delta v_k(\bar{\mathbf{g}} + \bar{g}_{j,k}, 0) = \Delta v_k(\hat{\bar{\mathbf{g}}}^{-E} + \bar{g}_{j,k}, 0)$ and therefore $\kappa > \Delta v_k(\hat{\bar{\mathbf{g}}}^{-E} + \bar{g}_{j,k}, 0)$ holds for any $\hat{\bar{\mathbf{g}}}^{-E} \subseteq \bar{\mathbf{g}}^{-E}$. From Lemma 4 it then again follows that, for γ sufficiently small, $\Delta v_k(\hat{\bar{\mathbf{g}}}^{-E} + \bar{g}_{j,k}, \gamma)$ is arbitrarily close to $\Delta v_k(\bar{\mathbf{g}} + \bar{g}_{j,k}, 0)$ and therefore $\kappa > \Delta v_k(\hat{\bar{\mathbf{g}}}^{-E} + \bar{g}_{j,k}, \gamma)$ for any $\hat{\bar{\mathbf{g}}}^{-E} \subseteq \bar{\mathbf{g}}^{-E}$ and γ sufficiently small. Finally, consider the last case, in which j is in the non-trivial component in $\bar{\mathbf{g}}$, while k is a singleton and $\gamma = 0$. From Lemma 5 and Lemma 6 we know that $x_j(\hat{\bar{\mathbf{g}}}^{-E}, 0) < x_j(\bar{\mathbf{g}}, 0)$, while $x_k(\hat{\bar{\mathbf{g}}}^{-E}, 0) = x_k(\bar{\mathbf{g}}, 0)$. One can then show by the same arguments as above that $\Delta v_j(\hat{\bar{\mathbf{g}}}^{-E} + \bar{g}_{j,k}, 0) < \Delta v_j(\bar{\mathbf{g}} + \bar{g}_{j,k}, 0)$ and $\Delta v_k(\hat{\bar{\mathbf{g}}}^{-E} + \bar{g}_{j,k}, 0) < \Delta v_k(\bar{\mathbf{g}} + \bar{g}_{j,k}, 0)$ and, since $\bar{\mathbf{g}}$ is a pairwise Nash equilibrium we know that either $\kappa \geq \Delta v_j(\bar{\mathbf{g}} + \bar{g}_{j,k}, 0)$ or $\kappa \geq \Delta v_k(\bar{\mathbf{g}} + \bar{g}_{j,k}, 0)$ holds and therefore $\kappa > \Delta v_j(\hat{\bar{\mathbf{g}}}^{-E} + \bar{g}_{j,k}, 0)$ or $\kappa > \Delta v_k(\hat{\bar{\mathbf{g}}}^{-E} + \bar{g}_{j,k}, 0)$. That is, creating the link between j and k is not profitable for at least one of the agents in any $\hat{\bar{\mathbf{g}}}^{-E} \subseteq \bar{\mathbf{g}}^{-E}$. It then follows again from Lemma 4 that j and k do not find it profitable to create a link in any $\hat{\bar{\mathbf{g}}}^{-E} \subseteq \bar{\mathbf{g}}^{-E}$ for γ sufficiently small. *Q.E.D.*

Lemma 10: *If γ is sufficiently small and $l \in D_i(\mathbf{g}'^{m}_i, \bar{\mathbf{g}})$ in configuration $(\mathbf{x}(\bar{\mathbf{g}}), \bar{\mathbf{g}})$, then creating the link $\bar{g}_{i,l} = 0$ is not profitable for agent i in any configuration $(\mathbf{x}(\hat{\bar{\mathbf{g}}}), \hat{\bar{\mathbf{g}}})$ with $\hat{\bar{\mathbf{g}}} \subseteq \bar{\mathbf{g}}'^{m}_i$.*

Proof. Assume first that $l \in D_i(\mathbf{g}'^{m}_i, \bar{\mathbf{g}})$ and $\mid D_i(\mathbf{g}'^{m}_i, \bar{\mathbf{g}}) \mid = 1$. Then $v(y(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})) - v(y(\bar{\mathbf{g}}) - x_l(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})) < \kappa$ must hold, as otherwise $\mathbf{g}'^{m}_i$ is not a minimal deletion best response. To see this, note that for any minimal best response other than $\mathbf{g}_i$, marginal deviation payoffs are strictly positive by Definition 3. Assume first that $\gamma = 0$. From Lemma 6 and $\hat{\bar{\mathbf{g}}} \subseteq \bar{\mathbf{g}}'^{m}_i$ it follows directly that $y(\hat{\bar{\mathbf{g}}}) + x_l(\bar{\mathbf{g}}) \leq y(\bar{\mathbf{g}})$. Next, define $\tilde{\bar{\mathbf{g}}} = \bar{\mathbf{g}} - \bar{g}_{i,l}$. From strategic complementarities it follows that $x_l(\bar{\mathbf{g}}) \geq x'_l(\tilde{\bar{\mathbf{g}}} + \bar{g}_{i,l})$ and from the arguments used in Lemma 9 that $x'_l(\tilde{\bar{\mathbf{g}}} + \bar{g}_{i,l}) \geq x'_l(\hat{\bar{\mathbf{g}}} + \bar{g}_{i,l})$ and therefore $x_l(\bar{\mathbf{g}}) \geq x'_l(\hat{\bar{\mathbf{g}}} + \bar{g}_{i,l})$. Note that then $y(\hat{\bar{\mathbf{g}}}) + x'_l(\hat{\bar{\mathbf{g}}} + \bar{g}_{i,l}) \leq y(\bar{\mathbf{g}})$ also holds. From the convexity of the value function and $\gamma = 0$ it then follows that $v(y(\hat{\bar{\mathbf{g}}}) + x'_l(\hat{\bar{\mathbf{g}}} + \bar{g}_{i,l}), z_i(\hat{\bar{\mathbf{g}}})) - v(y(\hat{\bar{\mathbf{g}}}), z_i(\hat{\bar{\mathbf{g}}})) \leq v(y(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})) - v(y(\bar{\mathbf{g}}) - x_l(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})) < \kappa$ holds for any $\hat{\bar{\mathbf{g}}}$ such that $\hat{\bar{\mathbf{g}}} \subseteq \bar{\mathbf{g}}'^{m}_i$. From Lemma 4 we know that $v(y(\hat{\bar{\mathbf{g}}}) + x'_l(\hat{\bar{\mathbf{g}}} + \bar{g}_{i,l}), z_i(\hat{\bar{\mathbf{g}}})) - v(y(\hat{\bar{\mathbf{g}}}), z_i(\hat{\bar{\mathbf{g}}})) < \kappa$ also holds for γ

sufficiently small. Therefore, agent i does not find it profitable to create a link with l in any configuration $(\mathbf{x}(\hat{\bar{\mathbf{g}}}), \hat{\bar{\mathbf{g}}})$ with $\hat{\bar{\mathbf{g}}} \subseteq \bar{\mathbf{g}}_i'^m$. Assume next that $l \in D_i(\mathbf{g}_i'^m, \bar{\mathbf{g}})$ and $\mid D_i(\mathbf{g}_i'^m, \bar{\mathbf{g}}) \mid > 1$. Then, for $\mathbf{g}_i'^m$ to be a minimal deletion best response, $v(y(\bar{\mathbf{g}}) - \sum_{j \in D_i(\mathbf{g}_i'^m, \bar{\mathbf{g}})} x_j(\bar{\mathbf{g}}) + x_l(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})) - v(y(\bar{\mathbf{g}}) - \sum_{j \in D_i(\mathbf{g}_i'^m, \bar{\mathbf{g}})} x_j(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})) \leq \kappa$ must hold. To see this, note that otherwise the deletion strategy $\mathbf{g}_i'^m + g_{i,l}$ yields a strictly higher deviation payoff than $\mathbf{g}_i'^m$ and $\mathbf{g}_i'^m$ is not a deletion best response. Assume first that $\gamma = 0$. Since $l \in D_i(\mathbf{g}_i'^m, \bar{\mathbf{g}})$, agent i and l are in a (non-trivial) connected component in $\bar{\mathbf{g}}$. We therefore know from Lemma 6 and $\hat{\bar{\mathbf{g}}} \subseteq \bar{\mathbf{g}}_i'^m$ that $y(\hat{\bar{\mathbf{g}}}) < y(\bar{\mathbf{g}}) - \sum_{j \in D_i(\mathbf{g}_i'^m, \bar{\mathbf{g}})} x_j(\bar{\mathbf{g}})$. Define $\tilde{\bar{\mathbf{g}}} = \bar{\mathbf{g}} - \sum_{j \in D_i(\mathbf{g}_i'^m, \bar{\mathbf{g}})} \bar{g}_{i,j}$ and note that from the above arguments it follows again that $x_l(\bar{\mathbf{g}}) \geq x_l'(\hat{\bar{\mathbf{g}}} + \bar{g}_{i,l})$. From the convexity of the value function and $\gamma = 0$ we know that $v(y(\hat{\bar{\mathbf{g}}}) + x_l'(\hat{\bar{\mathbf{g}}} + \bar{g}_{i,l}), z_i(\hat{\bar{\mathbf{g}}})) - v(y(\hat{\bar{\mathbf{g}}}), z_i(\hat{\bar{\mathbf{g}}})) < v(y(\bar{\mathbf{g}}) - \sum_{j \in D_i(\mathbf{g}_i'^m, \bar{\mathbf{g}})} x_j(\bar{\mathbf{g}}) + x_l(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})) - v(y(\bar{\mathbf{g}}) - \sum_{j \in D_i(\mathbf{g}_i'^m, \bar{\mathbf{g}})} x_j(\bar{\mathbf{g}}), z_i(\bar{\mathbf{g}})) \leq \kappa$ holds. Since the inequality in $v(y(\hat{\bar{\mathbf{g}}}) + x_l'(\hat{\bar{\mathbf{g}}} + \bar{g}_{i,l}), z_i(\hat{\bar{\mathbf{g}}})) - v(y(\hat{\bar{\mathbf{g}}}), z_i(\hat{\bar{\mathbf{g}}})) < \kappa$ is strict, it then follows again from Lemma 4 that the inequality also holds for γ sufficiently small. Therefore, agent i does not find it profitable to create a link with l in any configuration $(\mathbf{x}(\hat{\bar{\mathbf{g}}}), \hat{\bar{\mathbf{g}}})$ with $\hat{\bar{\mathbf{g}}} \subseteq \bar{\mathbf{g}}_i'^m$. *Q.E.D.*

Lemma 11: *If $\mathbf{g}_l'^m$ is a minimal deletion best response in configuration $(\mathbf{x}(\bar{\mathbf{g}}, \gamma), \bar{\mathbf{g}})$ for $\gamma = 0$, then $\mathbf{g}_l'^m$ is a minimal deletion best response in configuration $(\mathbf{x}(\bar{\mathbf{g}}, \gamma), \bar{\mathbf{g}})$ for γ sufficiently small.*

Proof. Recall that from Lemma 7 we know that a minimal deletion best response is unique, but there may be other deletion best responses. Note that, since any deletion deviation strategy that is not a deletion best response yields strictly lower deviation payoffs, and since equilibrium and deviation payoffs are continuous by Lemma 4, we only need to consider deletion best responses. Assume first that the minimal deletion best response, $\mathbf{g}_l'^m$, is the only deletion best response in $(\mathbf{x}(\bar{\mathbf{g}}, \gamma), \bar{\mathbf{g}})$ with $\gamma = 0$. From Lemma 4 it then follows directly that $\mathbf{g}_l'^m$ is also the minimal deletion best response in $(\mathbf{x}(\bar{\mathbf{g}}, \gamma), \bar{\mathbf{g}})$ for γ sufficiently small. Assume next that the minimal deletion best response, $\mathbf{g}_l'^m$, is not the only deletion best response in $(\mathbf{x}(\bar{\mathbf{g}}, \gamma), \bar{\mathbf{g}})$ for $\gamma = 0$. We distinguish two cases. Assume first that the number of links deleted in $\mathbf{g}_l'^m$ is the same as in any other deletion best response $\mathbf{g}_l'$. From Lemma 7 we know that then the set of neighbors after proposed deviations must be the same, i.e. $N_l(\bar{\mathbf{g}}_l'^m) = N_l(\bar{\mathbf{g}}_l')$ for any deletion best response $\mathbf{g}_l'$. Note that marginal payoffs of deletion best responses $\mathbf{g}_l'^m$ and $\mathbf{g}_l'$ are equal by assumption and can be written as follows. For $\mathbf{g}_l'^m$ they are given by

$$\tfrac{1}{(\beta+\gamma)} \left(\alpha + \lambda \textstyle\sum_{j \in N_l(\bar{\mathbf{g}}_l'^m)} x_j(\bar{\mathbf{g}}, \gamma) - \gamma z_l(\bar{\mathbf{g}}, \gamma)\right)^2 - \tfrac{1}{(\beta+\gamma)} \left(\alpha + \lambda \textstyle\sum_{j \in N_l(\bar{\mathbf{g}})} x_j(\bar{\mathbf{g}}, \gamma) - \gamma z_l(\bar{\mathbf{g}}, \gamma)\right)^2 + \left(\eta(\bar{\mathbf{g}}) - \eta(\mathbf{g}_l'^m)\right) \kappa,$$

while for $\mathbf{g}_l'$ they are given by

$$\tfrac{1}{(\beta+\gamma)} \left(\alpha + \lambda \textstyle\sum_{j \in N_l(\bar{\mathbf{g}}_l')} x_j(\bar{\mathbf{g}}, \gamma) - \gamma z_l(\bar{\mathbf{g}}, \gamma)\right)^2 - \tfrac{1}{(\beta+\gamma)} \left(\alpha + \lambda \textstyle\sum_{j \in N_l(\bar{\mathbf{g}})} x_j(\bar{\mathbf{g}}, \gamma) - \gamma z_l(\bar{\mathbf{g}}, \gamma)\right)^2 + \left(\eta(\bar{\mathbf{g}}) - \eta(\mathbf{g}_l')\right) \kappa.$$

The second term is equal in both expressions, while the third term is independent of γ, so that we can focus on the first term. Since $N_l(\bar{\mathbf{g}}_l'^m) = N_l(\bar{\mathbf{g}}_l')$, the marginal payoffs of $\mathbf{g}_l'^m$ and $\mathbf{g}_l'$ are also equal for $\gamma > 0$. That is, if $\mathbf{g}_l'^m$ is the minimal deletion best response in $(\mathbf{x}(\bar{\mathbf{g}}, \gamma), \bar{\mathbf{g}})$ for $\gamma = 0$, then $\mathbf{g}_l'^m$ is also the minimal deletion best response in $(\mathbf{x}(\bar{\mathbf{g}}, \gamma), \bar{\mathbf{g}})$ for γ sufficiently small. Assume next that there exists a deletion best response $\mathbf{g}_l'$ such that the number of links deleted in $\mathbf{g}_l'^m$ differs from $\mathbf{g}_l'$ in $(\mathbf{x}(\bar{\mathbf{g}}, \gamma), \bar{\mathbf{g}})$ for $\gamma = 0$. Note that by the above argument we can disregard any deletion best responses, such that the number of links deleted in $\mathbf{g}_l'^m$ is the same as in $\mathbf{g}_l'$ in $(\mathbf{x}(\bar{\mathbf{g}}, \gamma), \bar{\mathbf{g}})$ for $\gamma = 0$. We therefore consider deletion best responses $\mathbf{g}_l'$, such that $\mathbf{g}_l'^m \subset \mathbf{g}_l'$ and $| N_l(\bar{\mathbf{g}}_l'^m) |<| N_l(\bar{\mathbf{g}}_l') |$ holds. From Lemma 7 we know that then $N_l(\bar{\mathbf{g}}_l'^m) \subset N_l(\bar{\mathbf{g}}_l')$ also holds. We next show that for any such $\mathbf{g}_l'$, the marginal payoffs of $\mathbf{g}_l'^m$ are larger than for $\mathbf{g}_l'$ in $(\mathbf{x}(\bar{\mathbf{g}}, \gamma), \bar{\mathbf{g}})$ for $\gamma > 0$ sufficiently small. Note first that marginal payoffs for $\mathbf{g}_l'^m$ and $\mathbf{g}_l'$ are again given by the above expressions and equal for $\gamma = 0$ by definition. Since the second term is equal in both expressions, while the third term is independent of γ, we can again focus on the first term. Note next that Nash equilibrium effort levels are decreasing in γ. To see this, we write an agent's effort level for a given network $\bar{\mathbf{g}}$ as $x_i(\bar{\mathbf{g}}) = \alpha b_i(\bar{\mathbf{g}}, \frac{\lambda}{\beta}) / \left(\beta + \gamma b(\bar{\mathbf{g}}, \frac{\lambda}{\beta})\right)$ (by Theorem 1 in Ballester et al., 2006), where $b_i(\bar{\mathbf{g}}, \frac{\lambda}{\beta})$ is an agent's Bonacich centrality, given $\bar{\mathbf{g}}$ and $\frac{\lambda}{\beta}$, and $b(\bar{\mathbf{g}}, \frac{\lambda}{\beta})$ is the sum of agents' Bonacich centralities. Since $b_i(\bar{\mathbf{g}}, \frac{\lambda}{\beta})$ and $b(\bar{\mathbf{g}}, \frac{\lambda}{\beta})$ are independent of γ, Nash equilibrium effort levels are strictly decreasing in γ. From $N_l(\bar{\mathbf{g}}_l'^m) \subset N_l(\bar{\mathbf{g}}_l')$ it then follows directly that marginal deviation payoffs are strictly larger in $\mathbf{g}_l'^m$ than in $\mathbf{g}_l'$ for $\gamma > 0$. Therefore, $\mathbf{g}_l'^m$ is the minimal deletion deletion best response in $(\mathbf{x}(\bar{\mathbf{g}}, \gamma), \bar{\mathbf{g}})$ for γ sufficiently small. *Q.E.D.*

Proposition 2: *Assume $\bar{\mathbf{g}} \notin \{\bar{\mathbf{g}}^e\}$ and $e < t(\bar{\mathbf{g}})$. If γ is sufficiently small, then the optimal targeting policy prescribes eliminating a set of agents $E_i \in \mathscr{E}(e)$ for any value of δ.*

Proof. Recall that from Lemma 9 we know that after the elimination of a set of agents E, any link $\bar{g}_{j,k} = 0$ that is not present in the pairwise Nash equilibrium network $\bar{\mathbf{g}}$, is not profitable, neither in the configuration $(\mathbf{x}(\bar{\mathbf{g}}^{-E}), \bar{\mathbf{g}}^{-E})$, nor in any configuration $(\mathbf{x}(\hat{\bar{\mathbf{g}}}^{-E}), \hat{\bar{\mathbf{g}}}^{-E})$ with $\hat{\bar{\mathbf{g}}}^{-E} \subseteq \bar{\mathbf{g}}^{-E}$ for γ sufficiently small. Similarly, from Lemma 10 we know that if an agent i deletes a link to an agent l in a minimal deletion best response $\mathbf{g}_i'^m$ in configuration $(\mathbf{x}(\bar{\mathbf{g}}), \bar{\mathbf{g}})$, then agent i does not find profitable to create a link with l in any configuration $(\mathbf{x}(\hat{\bar{\mathbf{g}}}), \hat{\bar{\mathbf{g}}})$ with $\hat{\bar{\mathbf{g}}} \subseteq \bar{\mathbf{g}}_i'^m$ for γ sufficiently small. In time period $t = 0$ no links are created in $\bar{\mathbf{g}}_0^{-E}$ by Lemma 9 and therefore $\bar{\mathbf{g}}_1^{-E} = \bar{\mathbf{g}}_0^{-E}$. In time period $t = 1$ agents may delete links and therefore $\bar{\mathbf{g}}_2^{-E} \subseteq \bar{\mathbf{g}}_1^{-E}$. In time period $t = 2$, the creation of a link that was not in place in $t = 0$ is not profitable by Lemma 9, while the creation of any link that was deleted in $t = 1$ is not profitable by Lemma 10. Therefore, $\bar{\mathbf{g}}_3^{-E} = \bar{\mathbf{g}}_2^{-E}$. Applying the argument iteratively implies that no links are created in any time period t and we can disregard the creation of links. We next show that for any fixed number of eliminated agents e' with $e' \leq e < t(\bar{\mathbf{g}})$, it is optimal to eliminate a set of agents E_i such that $E_i \in \mathscr{E}(e')$ and in a second step that it is optimal to target $e' = e$ agents, i.e. to delete the maximum number of agents. Assume therefore that $E_i \in \mathscr{E}(e')$, $E_k \notin \mathscr{E}(e')$ and $\mid E_i \mid = \mid E_k \mid = e' \leq e$. From Lemma 3 we know that then $\bar{\mathbf{g}}_0^{-E_i} \subset \bar{\mathbf{g}}_0^{-E_k}$ holds, while from Theorem 2 in Ballester et al. (2006) we know that if $\bar{\mathbf{g}}_1 \subset \bar{\mathbf{g}}_2$, then $\sum_{j \in N} x(\bar{\mathbf{g}}_1) < \sum_{j \in N} x(\bar{\mathbf{g}}_2)$ holds. Therefore, $\sum_{j \in N(E_i)} x_j(\bar{\mathbf{g}}_0^{-E_i}) < \sum_{j \in N(E_k)} x_j(\bar{\mathbf{g}}_0^{-E_k})$ holds in time period $t = 0$. From the above we know that no links are created in $t = 0$ and therefore $\bar{\mathbf{g}}_1^{-E_i} = \bar{\mathbf{g}}_0^{-E_i}$ and $\bar{\mathbf{g}}_1^{-E_k} = \bar{\mathbf{g}}_0^{-E_k}$. Note that, since $\bar{\mathbf{g}}_1^{-E_i} \subset \bar{\mathbf{g}}_1^{-E_k}$, we know from Lemma 8 that the minimal best responses of an agent l in $\bar{\mathbf{g}}_1^{-E_i}$ and $\bar{\mathbf{g}}_1^{-E_k}$, given by $\mathbf{g}_1^{-E_i}{}'^m_l$ and $\mathbf{g}_1^{-E_k}{}'^m_l$, are such that $N_l(\bar{\mathbf{g}}_1^{-E_i}{}'^m_l) \subseteq N_l(\bar{\mathbf{g}}_1^{-E_k}{}'^m_l)$. Therefore, $\bar{\mathbf{g}}_2^{-E_i} \subseteq \bar{\mathbf{g}}_2^{-E_k}$ holds. Again, from the above we know that no links are created in $t = 2$. Applying the argument iteratively then yields $\bar{\mathbf{g}}_t^{-E_i} \subseteq \bar{\mathbf{g}}_t^{-E_k}$ $\forall t \geq 1$. From Theorem 2 in Ballester et al. (2006) it follows that $\sum_{j \in N(E_i)} x_j(\bar{\mathbf{g}}_t^{-E_i}) \leq \sum_{j \in N(E_k)} x_j(\bar{\mathbf{g}}_t^{-E_k})$ $\forall t \geq 1$, and therefore $\sum_{t=0}^{\infty} \delta^t \sum_{j \in N(E_i)} x_j(\bar{\mathbf{g}}_t^{-E_i}) < \sum_{t=0}^{\infty} \delta^t \sum_{j \in N(E_k)} x_j(\bar{\mathbf{g}}_t^{-E_k})$ holds for any $\delta \in (0, 1)$. Next we show that it is optimal to eliminate e agents and it is therefore optimal to target a set of agents $E_i \in \mathscr{E}(e)$. Take any set $E_k \in \mathscr{E}(e')$ with $e' < e < t(\bar{\mathbf{g}})$ and consider a corresponding second set $E_i \in \mathscr{E}(e)$ such that $E_k \subset E_i$. From the definition of $\mathscr{E}(\cdot)$ it is easy to see that such a set E_i always exists. Note that since the number of remaining agents are different when eliminating E_i and E_k, we cannot compare $\bar{\mathbf{g}}_t^{-E_i}$ and $\bar{\mathbf{g}}_t^{-E_k}$ directly. Consider therefore an intermediary network $\tilde{\bar{\mathbf{g}}}_0$, which is obtained from adding a set S of $\mid E_i \mid - \mid E_k \mid$ singletons to $\bar{\mathbf{g}}_0^{-E_i}$. We allocate the labels of agents in the set $E_i \setminus E_k$ to the agents in S. Note that then the set of agents in $\tilde{\bar{\mathbf{g}}}_0$ and $\bar{\mathbf{g}}_0^{-E_k}$ is the same, i.e. $N(\tilde{\bar{\mathbf{g}}}_0) = N(\bar{\mathbf{g}}_0^{-E_k})$. Moreover, $\tilde{\bar{\mathbf{g}}}_0 \subseteq \bar{\mathbf{g}}_0^{-E_k}$ holds. Recall from the above that for γ sufficiently small no links are created in $\bar{\mathbf{g}}_t^{-E_i}$, $\tilde{\bar{\mathbf{g}}}_t$ and $\bar{\mathbf{g}}_t^{-E_k}$ in any time period t. We next show that in any time period t, $\tilde{\bar{\mathbf{g}}}_t$ can be obtained from $\bar{\mathbf{g}}_t^{-E_i}$ by adding a set S of $\mid E_i \mid - \mid E_k \mid$ singletons. Assume first that $\gamma = 0$. By Lemma 9 no links are created in $t = 0$ and therefore $\bar{\mathbf{g}}_1^{-E_i} = \bar{\mathbf{g}}_0^{-E_i}$ and $\tilde{\bar{\mathbf{g}}}_1 = \tilde{\bar{\mathbf{g}}}_0$. Since $\gamma = 0$, incentives to delete links are the same in $\bar{\mathbf{g}}_1^{-E_i}$ and$\tilde{\bar{\mathbf{g}}}_1$ and minimal deletion best responses are such that $\tilde{g}_1{}'^m_{l,j} = g_1^{-E_i}{}'^m_{l,j}$ $\forall l, j \in N \setminus E_i$, while $\tilde{g}_1{}'^m_{l,j} = 0$ $\forall l, j : l \in N \setminus E_i$ and $j \in S$, and $\tilde{g}_1{}'^m_{l,j} = 0$ $\forall l \in S$. That is, for an agent $l \in N \setminus E_i$ the minimal deletion best response in $\tilde{\bar{\mathbf{g}}}_1$ can be obtained from the minimal deletion best response in $\bar{\mathbf{g}}_1^{-E_i}$ by adding zero entries for any agent in S, while minimal deletion best responses for agents in S in $\tilde{\bar{\mathbf{g}}}_1$ are zero vectors. Therefore, $\tilde{\bar{\mathbf{g}}}_2$ can be obtained from $\bar{\mathbf{g}}_2^{-E_i}$ by adding a set S of singleton agents. Since no links are created in any time period, we can use the previous argument iteratively. Therefore, $\tilde{\bar{\mathbf{g}}}_t$ can be obtained from $\bar{\mathbf{g}}_t^{-E_i}$ by adding a set S of singletons in any time period t. Note next that the sum of effort levels is always strictly smaller in $\bar{\mathbf{g}}_t^{-E_i}$ than in $\tilde{\bar{\mathbf{g}}}_t$. To see this, note that, since $\gamma = 0$, $x_j(\bar{\mathbf{g}}^{-E_i}) = x_j(\tilde{\bar{\mathbf{g}}})$ $\forall j \in N \setminus E_i$, while from the best response functions we know that for agents in S effort levels are strictly positive and given by $x_j(\tilde{\bar{\mathbf{g}}}) = \alpha/\beta > 0$ $\forall j \in S$. Therefore, $\sum_{t=0}^{\infty} \delta^t \sum_{j \in N(E_i)} x_j(\bar{\mathbf{g}}_t^{-E_i}) < \sum_{t=0}^{\infty} \delta^t \sum_{j \in N(E_k)} x_j(\tilde{\bar{\mathbf{g}}}_t)$. Since $\tilde{\bar{\mathbf{g}}}_0 \subseteq \bar{\mathbf{g}}_0^{-E_k}$ we know by Lemma 8, 9, 10 that $\tilde{\bar{\mathbf{g}}}_t \subseteq \bar{\mathbf{g}}_t^{-E_k}$ holds $\forall t$. Together with the former inequality it then follows that $\sum_{t=0}^{\infty} \delta^t \sum_{j \in N(E_i)} x_j(\bar{\mathbf{g}}_t^{-E_i}) < \sum_{t=0}^{\infty} \delta^t \sum_{j \in N(\bar{\mathbf{g}}_t^{-E_k})} x_j(\bar{\mathbf{g}}_t^{-E_k})$. It is therefore optimal to eliminate a set of agents E_i such that $E_i \in \mathscr{E}(e)$. Assume next that $\gamma > 0$ is sufficiently small. From Lemma 11 we know that if $\mathbf{g}_1^{-E_i}{}'^m_l$ is a minimal deletion best response in $(\mathbf{x}(\overline{\mathbf{g}}_1^{-E_i}), \overline{\mathbf{g}}_1^{-E_i})$ for $\gamma = 0$, then it is also a minimal deletion best response in in $(\mathbf{x}(\overline{\mathbf{g}}_1^{-E_i}), \overline{\mathbf{g}}_1^{-E_i})$ for γ sufficiently small. Likewise, if $\tilde{\mathbf{g}}_1{}'^m_l$ is a minimal deletion best response in $(\mathbf{x}(\tilde{\overline{\mathbf{g}}}_1), \tilde{\overline{\mathbf{g}}}_1)$ for $\gamma = 0$, then it is also a minimal deletion best response in in $(\mathbf{x}(\tilde{\overline{\mathbf{g}}}_1), \tilde{\overline{\mathbf{g}}}_1)$ for γ sufficiently small. That is, minimal deletion best responses are again the same in $\overline{\mathbf{g}}_1^{-E_i}$ and $\tilde{\overline{\mathbf{g}}}_1$ for $\gamma > 0$ sufficiently small. We can then use the same iterative argument as above to show that $\tilde{\overline{\mathbf{g}}}_t$ can be obtained from $\overline{\mathbf{g}}_t^{-E_i}$ by adding a set S of singletons in any time period t. Since the sum of effort levels is strictly smaller in $\overline{\mathbf{g}}_t^{-E_i}$ than in $\tilde{\overline{\mathbf{g}}}_t$ in any time period t for $\gamma = 0$, it follows from Lemma 4 that the sum of effort levels is also strictly smaller in $\overline{\mathbf{g}}_t^{-E_i}$ than in $\tilde{\overline{\mathbf{g}}}_t$ in any time period t for γ sufficiently small. Since $\tilde{\overline{\mathbf{g}}}_t \subseteq \overline{\mathbf{g}}_t^{-E_k}$ $\forall t$, $\sum_{t=0}^{\infty} \delta^t \sum_{j \in N(E_i)} x_j(\overline{\mathbf{g}}_t^{-E_i}) < \sum_{t=0}^{\infty} \delta^t \sum_{j \in N(E_k)} x_j(\overline{\mathbf{g}}_t^{-E_k})$ again holds for any $\delta \in (0,1)$ for $\gamma > 0$ sufficiently small. *Q.E.D.*

Definition 8: $\overline{k}_n^s = (\alpha^2\lambda(2\beta + (n-1)\lambda)) \,/\, (2(\beta+\lambda)(\beta+(n-1)\lambda)^2)$

$$\underline{k}_n^s = (\alpha^2\lambda(2\beta + 3\lambda)) \,/\, (2(\beta+\lambda)(\beta+(n-1)\lambda)^2)$$

$$k_{n-x}^e = \alpha^2\lambda(2\beta + 3\lambda) / \, (2(\beta+\lambda)(\beta+(n-x)\lambda)^2)$$

Lemma 12: *If $\gamma = \lambda$, then the star network is a pairwise Nash equilibrium network if and only if $k \in [\underline{k}_n^s, \overline{k}_n^s]$.*

Proof. This follows directly from the relevant bounds derived in Definition 8. *Q.E.D.*

Lemma 13: *Assume $(\mathbf{x}(\overline{\mathbf{g}}), \overline{\mathbf{g}})$ is a pairwise Nash equilibrium, $\overline{\mathbf{g}}$ is a star network, $\gamma = \lambda$ and $e = 1$. If $\overline{\mathbf{g}}_{\tilde{t}}^{-i}$ is complete (empty) for some $\tilde{t} \geq 0$, then $\overline{\mathbf{g}}_t^{-i}$ is complete (empty) for all $\overline{\mathbf{g}}_t^{-i}$ with $t \geq \tilde{t}$.*

Proof. We start by showing the statement for $\overline{\mathbf{g}}_{\tilde{t}}^{-i}$ complete. The average marginal benefit of being linked to $n-2$ agents in a complete network of $n-1$ agents is given by $\alpha^2\lambda(2\beta - (n-4)\lambda) / \, (2(\beta+\lambda)^3)$. One can show by algebraic manipulation that this expression is strictly larger than $\overline{k}_n^s$ for $\gamma = \lambda$.[35] Therefore, if $(\mathbf{x}(\overline{\mathbf{g}}), \overline{\mathbf{g}})$ is a pairwise Nash equilibrium, $\overline{\mathbf{g}}$ is a star network and $\overline{\mathbf{g}}_{\tilde{t}}^{-i}$ is complete for some $\tilde{t} \geq 0$, then $\overline{\mathbf{g}}_t^{-i}$ is complete for all $t \geq \tilde{t}$. Note next that from Definition 8 we know that for $\lambda = \gamma$ the marginal benefits of creating a link the empty network of $n-1$ agents are equal to $\underline{k}_n^s$. Since $(\mathbf{x}(\overline{\mathbf{g}}), \overline{\mathbf{g}})$ is a pairwise Nash equilibrium such that $\overline{\mathbf{g}}$ is a star network, we know that $k \geq \underline{k}_n^s$ must hold. Therefore, if $\overline{\mathbf{g}}_{\tilde{t}}^{-i}$ is empty for some $\tilde{t} \geq 0$, then $\overline{\mathbf{g}}_t^{-i}$ is empty for all $t \geq \tilde{t}$. *Q.E.D.*

[35] These calculations were conducted in Mathematica and are available from the author.

Proposition 4: *Assume $\overline{\mathbf{g}}$ is a pairwise Nash equilibrium, δ is sufficiently large and $\lambda = \gamma$, then the optimal targeting policy prescribes:*
i) if $e = 1$, eliminate the central agent;
ii) if $e = 2$ and $\kappa \in [\kappa^e_{n-2}, \overline{\kappa}^s_n]$, eliminate the center and a peripheral agent;
if $e = 2$, $\kappa \in [\underline{\kappa}^s_n, \kappa^e_{n-2})$ and $\lambda < \beta/(n^2 - 4n + 3)$, eliminate the central and a peripheral agent;
if $e = 2$, $\kappa \in [\underline{\kappa}^s_n, \kappa^e_{n-2})$ and $\lambda \geq \beta/(n^2 - 4n + 3)$, eliminating the central agent.

Proof. Assume first that $e = 1$. Denote the central agent with c and denote one peripheral player with p. If the planner eliminates agent c, then $\overline{\mathbf{g}}_0^{-c}$ is the empty network, while when the planner eliminates p, then $\overline{\mathbf{g}}_0^{-p}$ is a star of $n - 1$ agents. Therefore, $\overline{\mathbf{g}}_0^{-c} \subset \overline{\mathbf{g}}_0^{-p}$ and, by Theorem 2 in Ballester et al. (2006), we know that then the sum of Nash equilibrium effort levels is strictly lower in $\overline{\mathbf{g}}_0^{-c}$ than in $\overline{\mathbf{g}}_0^{-p}$. From Lemma 13 we know that $\overline{\mathbf{g}}_t^{-c}$ is also empty for all $t \geq 1$. Therefore, $\overline{\mathbf{g}}_t^{-c} \subseteq \overline{\mathbf{g}}_t^{-p}$ for all $t \geq 1$ and $\sum_{t=0}^{\infty} \delta^t \sum_{j \in N \setminus \{c\}} x_j(\overline{\mathbf{g}}_t^{-c}) < \sum_{t=0}^{\infty} \delta^t \sum_{j \in N \setminus \{p\}} x_j(\overline{\mathbf{g}}_t^{-p})$. Next we compare eliminating a central agent with not intervening. Note first that the sum of effort levels in the empty network of n agents is given by $n\alpha/(\beta + n\gamma)$ and the first derivative with respect to n is given by $\alpha\beta/(\beta + n\gamma)^2 > 0$. Note next that $\overline{\mathbf{g}}_t^{-c}$ is the empty network of $n - 1$ agents in, while $\overline{\mathbf{g}}^e$ is the empty network of n agents, so that $\sum_{j \in N \setminus c} x_j(\overline{\mathbf{g}}_t^{-c}) < \sum_{j \in N} x_j(\overline{\mathbf{g}}^e)$ holds. From $\sum_{j \in N} x_j(\overline{\mathbf{g}}^e) \leq \sum_{j \in N} x_j(\overline{\mathbf{g}})$ $\forall \overline{\mathbf{g}}$ it then follows that the optimal targeting policy prescribes eliminating the central agent. Assume next that $e = 2$ and $\kappa \in [\kappa^e_{n-2}, \overline{\kappa}^s_n]$. That is, in an empty network of $n - 2$ agents no new links will be created. From an argument analogous to the one above, one can show that the optimal targeting policy prescribes eliminating the central and a peripheral agent. Assume next that $e = 2$ and $\kappa \in [\underline{\kappa}^s_n, \kappa^e_{n-2})$. Denote a second peripheral player with p'. Note first that from the expressions for κ^e_{n-2} and $\underline{\kappa}^s_{n-2}$ it follows directly that $\kappa^e_{n-2} < \underline{\kappa}^s_{n-2}$. That is, if $\kappa \in [\underline{\kappa}^s_n, \kappa^e_{n-2})$, then for $E = \{c, p\}$ and $E' = \{p, p'\}$ we have that the network is complete for all $t \geq 1$. Since in $t = 0$ for $E = \{c, p\}$ the network is empty, while for $E' = \{p, p'\}$ it is a star network, the planner prefers $E = \{c, p\}$ over $E' = \{p, p'\}$. That is, we know that $E = \{p, p'\}$ is not an optimal targeting policy. From the above it follows that if $e = 1$, then eliminating the central agent is optimal. We next compare eliminating $E = \{c, p\}$ and $E' = \{c\}$. If $E = \{c, p\}$, then the network is the empty network of $n - 2$ agents in $t = 0$ and the complete network of $n - 2$ agents in $t = 1$. One can show via algebraic manipulation (and using $\beta > (n - 1)\lambda$) that the average marginal benefits of keeping $n - 3$ links in the complete network of $n - 2$ agents is strictly larger than $\overline{\kappa}^s_n$ and the network therefore remains complete for all $t \geq 1$. In contrast, if $E' = \{c\}$, then for all time periods, the network is the empty network of $n - 1$ agents. The sum of payoffs in the complete network of $n - 2$ agents is given by $(n - 2)\alpha/(\beta + \lambda)$, while for the empty network of $n - 1$ agents it is given by $(n - 1)\alpha/(\beta + (n - 1)\lambda)$. The latter is weakly larger than the former if and only if $\lambda \leq \beta/(n^2 - 4n + 3)$ and ii) follows directly from the assumption that δ is sufficiently large. *Q.E.D.*

7 References

Ahuja, G., (2000), Collaboration networks, structural holes, and innovation: A longitudinal study, Administrative Science Quarterly 45, 425–455.

Bala, V. and S. Goyal (2000), A Non-Cooperative Model of Network Formation, Econometrica, Volume68, Issue 5, 1181-1229.

Ballester, C., Calvó-Armengol, A. and Y. Zenou (2006), Who's who in networks. Wanted: the key player, Econometrica 74, 1403-1417.

Ballester, C., Zenou, Y., Calvó-Armengol, A. (2010), Delinquent networks, Journal of the European Economic Association, 8(1), 34-61.

Bätz, O. (2015), Social Activity and Network Formation, Theoretical Economics, 10 (2015), 315–340.

Belhaj, M. and F. Deroian, (2019), Group Targeting under Networked Synergies, mimeo.

Bloch, F. and N. Querou, (2013), Pricing in Social Networks, Games and Economic Behavior, 80, 263–281.

Bonacich, P., (1987), Power and Centrality: A Family of Measures, American Journal of Sociology, 92, 1170-1182.

Borgatti, S., (2006), Identifying Sets of Key Players in a Social Network, Computational and Mathematical Organization Theory, 12, 21–34.

Canter, D., (2004), A Partial Order Scalogram Analysis of Criminal Network Structures, Behaviormetrika, 31, No.2, 131-152.

Calvó-Armengol, A., Patacchini, E., Zenou, Y. (2009), Peer effects and social networks in education, The Review of Economic Studies, 76(4), 1239-1267.

D'Aspremont, C. and Jacquemin, A., (1988), Cooperative and noncooperative R&D in duopoly with spillovers, The American Economic Review, 78(5):1133–1137.

Demange, G., (2017), Optimal Targeting Strategies in a Network under Complementarities, Games and Economic Behavior, 105, pp. 84-103.

Dorn, N., N. South, (1990), Drug Markets and Law enforcements, British Journal of Criminology, 30 (2), 171-188.

Dorn, N., K. Murji, N. South, (1992), Traffickers: Drug markets and drug enforcement, London: Routledge.

Fainmesser, I. and A. Galeotti (2017), Pricing Network Effects, Review of Economic Studies, 83, 165–198.

Galeotti, A., Golub, B., Goyal, S., (2020), Targeting Interventions in Networks, Econometrica, forthcoming.

Galeotti, A. and S. Goyal, (2009), Influencing the Influencers: A Theory of Strategic Diffusion, The Rand Journal of Economics, 40, 509–532.

Galeotti, A. and S. Goyal, (2010), The Law of the Few, The American Economic Review, 100, No. 4, 1468-92.

Galeotti, A. and B. W. Rogers, (2013), Strategic Immunization and Group Structure, American Economic Journal: Microeconomics, 5, 1–32.

Goyal, S., and S. Joshi, (2003), Networks of Collaboration in Oligopoly, Games and Economic Behavior 43, 57-85.

Goyal, S. and Moraga-Gonzalez, J. L., (2001), R&D networks. RAND Journal of Economics, 32(4):686–707.

Hagedoorn, J., (2002), Inter-firm R&D partnerships: an overview of major trends and patterns since 1960, Research Policy 31 (4), 477–492.

Hauck, R.V., Atabakhsh, H., Ongvasith, P., Gupta, H. and H. Chen, (2002), Using Coplink to analyze criminal-justice data, IEEE Computer 35, 3: 3037.

Haynie, D.L., (2001), Delinquent peers revisited: Does network structure matter?, American Journal of Sociology, 106, 1013-1057.

Helsley, R. and Y. Zenou, (2014), Social networks and interactions in cities, Journal of Economic Theory, 150, 426-466.

Hiller, T., (2017), Peer Effects in Endogenous Networks, Games and Economic Behavior, Vol. 105, 349-367.

Hiller, T., (2020), A Simple Model of Network Formation with Congestion Effects, mimeo.

Jackson, M. O. (2010), Social and economic networks, Princeton University Press.

Jackson, M. O., B. Rogers and Y. Zenou, (2017), The Economic Consequences of Social-Network Structure, Journal of Economic Literature, 55(1), 49-95.

Jackson, M. O. and A. Watts (2002a), On the formation of interaction networks in social coordination games, Games and Economic Behavior, 41, 265-291.

Jackson, M. O. and A. Watts (2002b), The evolution of interaction networks in social and economic networks, Journal of Economic Theory, 106, 265-296.

Johnston L. (2000), The Social Structure of Football Hooliganism. In D. Canter and L.J. Alision (Eds.), The Social Psychology of Crime, Aldershot, Dartmouth, pp. 153-188.

Joshi S., A. S. Mahmud, (2016), Network Formation under Multiple Sources of Externalities, Journal of Public Economic Theory, 18 (2), 2016, 148-167.

Kinateder, M., and Merlino, L. P. (2017), Public goods in endogenous networks, American Economic Journal: Microeconomics, 9(3), 187-212.

Kitsak, M., Riccaboni, M., Havlin, S., Pammolli, F., and Stanley, H., (2010), Scale-free models for the structure of business firm networks, Physical Review E, 81:036117.

König, M. (2016), Dynamic R&D Networks with Process and Product Innovations, mimeo.

König, M. D., Hsie, C.-S., Liu, X., (2018), Network Formation with Local Complements and Global Substitutes: The Case of R&D Networks, mimeo.

König, M., C. J. Tessone, and Y. Zenou, (2014), Nestedness in Networks: A Theoretical Model and Some Applications, Theoretical Economics, 9 (3), 695–752.

König M., X. Liu, Y. Zenou, (2019), R&D Networks: Theory, Empirics and Policy Implications, Review of Economics and Statistics, Volume 101, Issue 3, p.476-491.

Mastrobuoni, G. and E. Patacchini, (2012), Organized Crime Networks: an Application of Network Analysis Techniques to the American Mafia, Review of Network Economics 11(3), Article 10, 1-41.

Monderer, D., L. Shapley, (1996), Potential Games, Games and Economic Behavior, Vol. 14, 124-143.

Patacchini, E. and Y. Zenou, (2012), Juvenile delinquency and conformism, Journal of Law, Economics, and Organization 28, 1-31.

Powell, W. W., White, D. R., Koput, K. W., Owen-Smith, J., (2005), Network dynamics and field evolution: The growth of interorganizational collaboration in the life sciences, American Journal of Sociology, 110 (4), 1132–1205.

Riccaboni, M., Pammolli, F., (2002), On firm growth in networks, Research Policy 31 (8-9), 1405– 1416.

Roijakkers, N., Hagedoorn, J., (2006), Inter-firm R&D partnering in pharmaceutical biotechnology since 1975: Trends, patterns, and networks, Research Policy, 35 (3), 431–446.

Rosenkopf, L. and Schilling, M., (2007), Comparing alliance network structure across industries: Observations and explanations, Strategic Entrepreneurship Journal, 1:191–209.

Ruggiero V., N. South, (1995), Eurodrugs: Drug use, markets and trafficking in Europe, London, UCL Press.

Sarnecki, J., (2001), Delinquent Networks: Youth Co-Offending in Stockholm, Cambridge: Cambridge University Press.

Tayebi, M.A., Bakker, L., Glässer, U. and V. Dabbaghian (2011), Organized crime structures in co-offending networks, IEEE Ninth International Conference on Autonomic and Secure Computing (DASC), pp. 846-853.

Tomasello, M. V. , Napoletano, M. Garas, A., Schweitzer, F., (2017), The rise and fall of R&D networks, Industrial and Corporate Change, Volume 26, Issue 4, pp. 617–646.

Tremblay, R.E., Masse, L., Pagani, L. and F. Vitaro, (1996), From childhood physical aggression to adolescent maladjustment: The Montreal Prevention Experiment, In R.D. Peters and R.J. McMahon (Eds.), Preventing Childhood Disorders, Substance Abuse, and Delinquency, Thousand Oaks, CA: Sage Publications.

Valente, T., (2012), Network Interventions, Science, 337, 49–53.

Vega-Redondo, F., (2007), Complex social networks, No. 44, Cambridge University Press.

Zenou Y., (2016), Key players. The Oxford Handbook of the Economics of Networks. Mar 1; 11.

8 ONLINE APPENDIX: TARGETING WHEN THE NETWORK IS ENDOGENOUS

Timo Hiller

Abstract

We first formally relate the payoff function in Ballester et al. (2006) to the one presented in the main part of the paper. Then we provide the proof of Proposition 3 and introduce the one-sided network formation model.

8.1 The relationship with Ballester et al. (2006)

Ballester et al. (2006) start by assuming the utility function below

$$u_i(x_1, ..., x_n) = \alpha x_i + \tfrac{1}{2}\sigma x_i^2 + \textstyle\sum_{j \neq i} \sigma_{ij} x_i x_j,$$

with $\alpha > 0$ and $\sigma < 0$. By defining $\underline{\sigma} = min\{\sigma_{ij} \mid i \neq j\}$ and $\overline{\sigma} = max\{\sigma_{ij} \mid i \neq j\}$, the utility functions is rewritten as follows

$$u_i(\mathbf{x}) = \alpha x_i - \tfrac{1}{2}(\beta - \gamma)x_i^2 - \gamma x_i \textstyle\sum_{j \in N} x_j + \lambda \sum_{j \in N} g_{ij} x_i x_j,$$

where $\gamma = -min\{\underline{\sigma}, 0\} \geq 0$, $\lambda = \overline{\sigma} + \gamma \geq 0$. The authors assume $\lambda > 0$ (and thereby rule out the case when $\underline{\sigma} = \overline{\sigma}$) and interpret $g_{ij} = (\sigma_{ij} + \gamma)/\lambda$ as a directed link from i to j in network $\mathbf{g}$. Note that $u_i(\mathbf{x})$ can then be rearranged and rewritten as

$$u_i(\mathbf{x}) = \alpha x_i - \tfrac{1}{2}(\beta + \gamma)x_i^2 - \gamma x_i \textstyle\sum_{j \in N \setminus \{i\}} x_j + \lambda \sum_{j \in N} g_{ij} x_i x_j.$$

When $\sigma_{ij} \in \{\underline{\sigma}, \overline{\sigma}\}$ for all $i \neq j$, then adjacency matrix corresponding to $\mathbf{g}$ is a symmetric $(0, 1)$-matrix and $\mathbf{g}$ is undirected and unweighted. We further assume that $\lambda \geq \gamma$, which yields strategic complementarities in effort levels for connected agents. In the context of Ballester et al. (2006) this corresponds to $\overline{\sigma} \geq 0$. Finally, note that when $\overline{\sigma} = 0$, then $\lambda = \gamma > 0$.

8.2 Proof of Proposition 3

In Proposition 3 we present a sufficient condition for a pairwise Nash equilibrium to exists, such that the optimal targeting policy prescribes eliminating an agent with the lowest number of links and lowest effort level. It is sufficient that $\lambda - \gamma \geq 0$ is sufficiently small (or put differently, that γ is sufficiently large, given our restrictions on parameters), δ is sufficiently large and that linking cost κ is between two bounds, $\underline{\kappa}'$ and $\overline{\kappa}'$, defined below. Note also that for the following we assume that $n \geq 5$.

Below we define the bounds on linking cost used in Proposition 3.

Definition 9:

$$\kappa_1' = (\alpha^2\beta\lambda(2\beta^2 - (2n-9)\beta\lambda - 2(n-3)\lambda^2) \,/\, (2(\beta+\lambda)(\beta^2 + 2\beta\alpha - (n-3)\alpha)^2)$$
$$\kappa_2' = (\alpha^2\beta\lambda(2\beta^2 - (2n-11)\beta\lambda - 2(n-4)\lambda^2)) \,/\, (2(\beta+\lambda)\beta^2 + 4\beta\lambda - 3(n-4)\lambda^2)^2)$$

Proposition 3: *Assume $e = 1$. If $\lambda - \gamma \geq 0$ is sufficiently small, δ is sufficiently large and $\kappa_1' < \kappa < \kappa_2'$, then there exists a pairwise Nash equilibrium, such that the optimal targeting policy prescribes eliminating an agent with the fewest links.*

Proof. For the following, take $\bar{\mathbf{g}}$ to be a dominant group network with a core of size $c = n-2$. Denote by $\bar{\mathbf{g}}_0^{-c}$ the network when a core agent is eliminated from $\bar{\mathbf{g}}$ and denote by $\bar{\mathbf{g}}_0^{-p}$ the network when a periphery agent is eliminated from $\bar{\mathbf{g}}$. The corresponding Nash equilibrium effort levels of agents in the core are denoted by $x_c(\bar{\mathbf{g}})$ and in the periphery by $x_p(\bar{\mathbf{g}})$. Deviation effort levels are written in the following way. For example, $x_c'(\bar{\mathbf{g}}_0^{-p} + \bar{g}_{c,p})$ is the deviation effort level of a core agent in configuration $(\mathbf{x}(\bar{\mathbf{g}}_0^{-p}), \bar{\mathbf{g}}_0^{-p})$, when creating a link to an agent in the periphery. The remaining deviation effort levels are defined analogously. We start by assuming that $\lambda = \gamma$. Consider the following two conditions, (1) and (2), which ensure that $(\mathbf{x}(\bar{\mathbf{g}}), \bar{\mathbf{g}})$ is a pairwise Nash equilibrium.

$$(1)\ (v((c-1)x_c(\bar{\mathbf{g}}), (c-1)x_c(\bar{\mathbf{g}}) + 2x_p(\bar{\mathbf{g}})) - v(0, (c-1)x_c(\bar{\mathbf{g}}) + 2x_p(\bar{\mathbf{g}}))) \,/\, (c-1) =$$
$$= \frac{\alpha^2\beta^2\lambda(2\beta+2\gamma+\lambda-n\lambda)}{2(\beta+\gamma)(\beta(\beta+(c+n)\gamma)-(n-1)(\beta+c\gamma)\lambda)^2} > \kappa$$

$$(2)\ v(x_c'(\bar{\mathbf{g}} + \bar{g}_{c,p}), (c-1)x_c(\bar{\mathbf{g}}) + x_c'(\bar{\mathbf{g}} + \bar{g}_{c,p}) + x_p(\bar{\mathbf{g}})) - v(0, cx_c(\bar{\mathbf{g}}) + x_p(\bar{\mathbf{g}})) =$$
$$= \frac{\alpha^2(\beta+\gamma)\lambda(\beta^2+\beta\lambda+(n-1)(\gamma-\lambda)\lambda)(2\beta^3-(n-1)(3\gamma-\lambda)\lambda^2-\beta\lambda(4(n-2)\gamma+\lambda)+\beta^2(4\gamma+(3-2n)\lambda))}{2(\beta+2\gamma-\lambda)^2(\beta+\lambda)^2(\beta(\beta+(n+c)\gamma)-(n-1)(\beta+c\gamma)\lambda)^2} < \kappa$$

More precisely, the left hand side of condition (1) is given by the average marginal benefits per link of an agent in the core deleting all links. Since the value function is convex, condition (1) is sufficient for there to be no profitable deviation that involves deleting links in $(\mathbf{x}(\bar{\mathbf{g}}), \bar{\mathbf{g}})$. The left hand side of condition (2) is given by the marginal benefit of an agent in the periphery linking to an agent in the core. Note that from the arguments used Lemma 9 it follows directly that, if an agent in the periphery does not find it profitable to an agent in the core, then two agents in the periphery do not find it profitable to create a link. That is, condition (2) ensures that there is not profitable deviation that involves creating a link. Therefore, if conditions (1) and (2) hold, $(\mathbf{x}(\bar{\mathbf{g}}), \bar{\mathbf{g}})$ is a pairwise Nash equilibrium. Conditions (3) and (4) for $(\mathbf{x}(\bar{\mathbf{g}}_0^{-p}), \bar{\mathbf{g}}_0^{-p})$ below are analogous to condition (1) and (2). That is, if conditions (3) and (4) hold, then the configuration after eliminating an agent in the periphery, $(\mathbf{x}(\bar{\mathbf{g}}_0^{-p}), \bar{\mathbf{g}}_0^{-p})$, is a pairwise Nash equilibrium.

(3) $(v((c-1)x_c(\bar{\mathbf{g}}^{-p}), (c-1)x_c(\bar{\mathbf{g}}^{-p}) + x_p(\bar{\mathbf{g}}^{-p})) - v(0, (c-1)x_c(\bar{\mathbf{g}}) + x_p(\bar{\mathbf{g}})))\,/\,(c-1) =$

$$= \frac{\alpha^2\beta^2\lambda(2\beta+2\gamma+2\lambda-\lambda n)}{2(\beta+\gamma)(\beta(\beta+(c+n-1)\gamma)-(n-2)(\beta+c\gamma)\lambda)^2} > \kappa$$

(4) $v(x'_c(\bar{\mathbf{g}}^{-p} + \bar{g}_{c,p}), (c-1)x_c(\bar{\mathbf{g}}^{-p}) + x'_c(\bar{\mathbf{g}}^{-p} + \bar{g}_{c,p})) - v(0, cx_c(\bar{\mathbf{g}})) =$

$$= \frac{\alpha^2(\beta+\gamma)\lambda(\beta^2+\beta\lambda+(n-2)(\gamma-\lambda)\lambda)(2\beta^3-(n-2)(3\gamma-\lambda)\lambda^2-\beta\lambda(4(n-3)\gamma+\lambda)+\beta^2(4\gamma+(5-2n)\lambda))}{2(\beta+2\gamma-\lambda)^2(\beta+\lambda)^2(\beta(\beta+(n+c-1)\gamma)-(n-2)(\beta+c\gamma)\lambda)^2} < \kappa$$

The left hand side of condition (5) yields the marginal payoffs of an agent in the periphery creating a link with another agent in the periphery in $(\mathbf{x}(\bar{\mathbf{g}}^{-c}), \bar{\mathbf{g}}^{-c})$. From the arguments in Lemma 9 it again follows directly that then also all links between agents in the core and the periphery in $(\mathbf{x}(\bar{\mathbf{g}}_0^{-c}), \bar{\mathbf{g}}_0^{-c})$ are profitable and $\bar{\mathbf{g}}_1^{-c}$ is therefore the complete network of $n-1$ agents. Finally, the left hand side condition (6) is given by the average marginal payoff per link of an agent in configuration $(\mathbf{x}(\bar{\mathbf{g}}_1^{-c}), \bar{\mathbf{g}}_1^{-c})$ (where $\bar{\mathbf{g}}_1^{-c}$ is the complete network of $n-1$ agents). That is, $(\mathbf{x}(\bar{\mathbf{g}}_1^{-c}), \bar{\mathbf{g}}_1^{-c})$ is a pairwise Nash equilibrium.

(5) $v(x'_p(\bar{\mathbf{g}}^{-c} + \bar{g}_{p,p}), (c-1)x_c(\bar{\mathbf{g}}^{-c}) + x'_p(\bar{\mathbf{g}}^{-c} + \bar{g}_{p,p})) - v(0, (c-1)x_c(\bar{\mathbf{g}}^{-c})) =$

$$= \frac{\alpha^2(\beta+\gamma)(\beta+(n-1)(\gamma-\lambda))\lambda(2\beta^2-(n-1)(3\gamma-\lambda)+\beta(4\gamma+\lambda-2n\lambda))}{2(\beta+2\gamma-\lambda)^2(\beta+(n+c-1)\gamma)-(n-1)(\beta+(c-1)\gamma)\lambda)^2} > \kappa$$

(6) $\frac{v((n-2)x(\bar{\mathbf{g}}_1^{-c}),(n-2)x(\bar{\mathbf{g}}_1^{-c}))-v(0,(n-2)x(\bar{\mathbf{g}}_1^{-c}))}{n-2} =$

$$= \frac{\alpha^2\lambda(2(\beta+\gamma)-(n-2)\lambda)}{2(\beta+\gamma)(\beta+(n-1)\gamma-(n-2)\lambda)^2} > \kappa$$

One can show that the above conditions hold simultaneously with strict inequality for $\lambda = \gamma > 0$ and $\kappa'_1 < \kappa < \kappa'_2$.[36] That is, if the above conditions hold, then $\bar{\mathbf{g}}$ is a pairwise Nash equilibrium, and $\bar{\mathbf{g}}_t^{-p} = \bar{\mathbf{g}}^{-p}\ \forall t \geq 0$, while $\bar{\mathbf{g}}_0^{-c} = \bar{\mathbf{g}}^{-c}$ and $\bar{\mathbf{g}}_t^{-c} = \bar{\mathbf{g}}^{c}\ \forall t \geq 1$. That is, $\bar{\mathbf{g}}_0^{-c} \subset \bar{\mathbf{g}}_0^{-p}$ by Lemma 3, while $\bar{\mathbf{g}}_t^{-p} \subset \bar{\mathbf{g}}_t^{-c}\ \forall t \geq 1$. By Proposition 2 of Ballester et al. (2006) $\sum_{j\in N\backslash p} x_j(\bar{\mathbf{g}}_t^{-p}) > \sum_{j\in N\backslash c} x_j(\bar{\mathbf{g}}_t^{-c})$ for $t = 0$ and $\sum_{j\in N\backslash p} x_j(\bar{\mathbf{g}}_t^{-p}) < \sum_{j\in N\backslash c} x_j(\bar{\mathbf{g}}_t^{-c})\ \forall t \geq 1$. Therefore, $\sum_{t=0}^{\infty} \delta^t \sum_{j\in N\backslash p} x_j(\bar{\mathbf{g}}_t^{-p}) < \sum_{t=0}^{\infty} \delta^t \sum_{j\in N\backslash c} x_j(\bar{\mathbf{g}}_t^{-c})$ for δ sufficiently close to 1. Note next that since the relevant conditions hold with strict inequality, we know from Lemma 4 that the above statement also holds for $\lambda - \gamma \geq 0$ sufficiently small. *Q.E.D.*

[36] The calculations were executed in Mathematica and are available from the author upon request.

8.3 One-sided Link Formation and Convergence

We assume that network formation is one-sided with two-sided flow. That is, it is sufficient for only one agent to extend a link in order for both agents involved in the link to benefit from each other's effort level. This specification allows us to use Nash equilibrium as a solution concept. Moreover, one can show that the payoff function admits a potential. We can therefore rely on Monderer and Shapley (1996) to ensure convergence of best response dynamics. Note that Nash equilibrium allows for deviations in which agents create multiple links unilaterally and may simultaneously delete any subset of their existing links. However, only the deviating agent is assumed to adjust effort levels.

Let $N = \{1, 2, ..., n\}$ be the set of players with $n \geq 3$. As before, each player i chooses a personal effort level $x_i \in X$ and a set of links, which are represented as a row vector $\mathbf{g}_i = (g_{i,1}, ..., g_{ii-1}, g_{ii+1}, ..., g_{in})$, where $g_{ij} \in \{0, 1\}$ for each $j \in N \backslash \{i\}$. Assume $X = [0, +\infty)$ and $\mathbf{g}_i \in G_i = \{0, 1\}^{n-1}$. The set of strategies of i is denoted by $S_i = X \times G_i$ and the set of strategies of all players by $S = S_1 \times S_2 \times ... \times S_n$. A strategy profile $\mathbf{s} = (\mathbf{x}, \mathbf{g}) \in S$ specifies the individual effort level of each player, $\mathbf{x} = (x_1, x_2, ..., x_n)$, and a set of links $\mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_n)$. Agent i is said to sustain or extend a link to j, if $g_{i,j} = 1$ and to receive a link from j, if $g_{j,i} = 1$. The network of relations $\mathbf{g}$ is a directed graph, i.e. it is possible that $g_{i,j} \neq g_{j,i}$. Let $N_i(\mathbf{g}) = \{j \in N : g_{i,j} = 1\}$ be the set of agents i has extended a link to and define $\eta_i(\mathbf{g}) = |N_i(\mathbf{g})|$. Call the closure of $\mathbf{g}$ an undirected network, denoted by $\bar{\mathbf{g}} = cl(\mathbf{g})$, where $\bar{g}_{i,j} = \max\{g_{i,j}, g_{j,i}\}$ for each i and j in N. Denote with $N_i(\bar{\mathbf{g}}) = \{j \in N : \bar{g}_{i,j} = 1\}$ the set of players that are directly connected to i. As before, we write $y_i = \sum_{j \in N_i(\bar{\mathbf{g}})} x_j$ for the effort level of i's direct neighbors. The aggregate effort level of all agents other than i is written as $z_i(\bar{\mathbf{g}}) = \sum_{j \in N \backslash \{i\}} x_j$. We will drop the subscript of y_i when it is clear from the context.

Payoffs of player i under strategy profile $\mathbf{s} = (\mathbf{x}, \mathbf{g})$ are given by

$$\Pi_i(\mathbf{s}) = \pi_i(\mathbf{x}, \mathbf{g}) - \eta_i(\mathbf{g})k,$$

where k denotes the cost of extending a link. Gross payoffs, i.e. payoffs excluding linking cost, $\pi_i(\mathbf{x}, \bar{\mathbf{g}})$, are again given by the linear-quadratic payoff function with local complementarities and global substitutes (Ballester et al., 2006). That is,

$$\pi_i(\mathbf{x}, \bar{\mathbf{g}}) = \alpha x_i - \tfrac{1}{2}(\beta + \gamma)x_i^2 + \lambda x_i \textstyle\sum_{j \in N_i(\bar{\mathbf{g}})} x_j - \gamma x_i \sum_{j \in N \backslash \{i\}} x_j \ \forall i \in N.$$

A Nash equilibrium is a strategy profile $\mathbf{s} = (\mathbf{x}, \mathbf{g})$ such that

$$\Pi_i(\mathbf{s}_i, \mathbf{s}_{-i}) \geq \Pi_i(\mathbf{s}'_i, \mathbf{s}_{-i}), \forall \mathbf{s}'_i \in S_i, \forall i \in N.$$

We again assume that $\beta > (n - 1)\lambda$ to guarantee existence on a fixed network and, in order to guarantee convergence, make the additional assumption that $2\lambda/(\beta - \gamma) < 1/(n - 1)$, as explained in more detailed below. The best response and value functions are as in the main part of the manuscript. One can show that in any Nash equilibrium the undirected is a nested split graph. Since z_i is fixed in any deviation considered, we can use the same arguments as the ones presented in the online appendix to Hiller (2017). Formal proofs are available from the author.

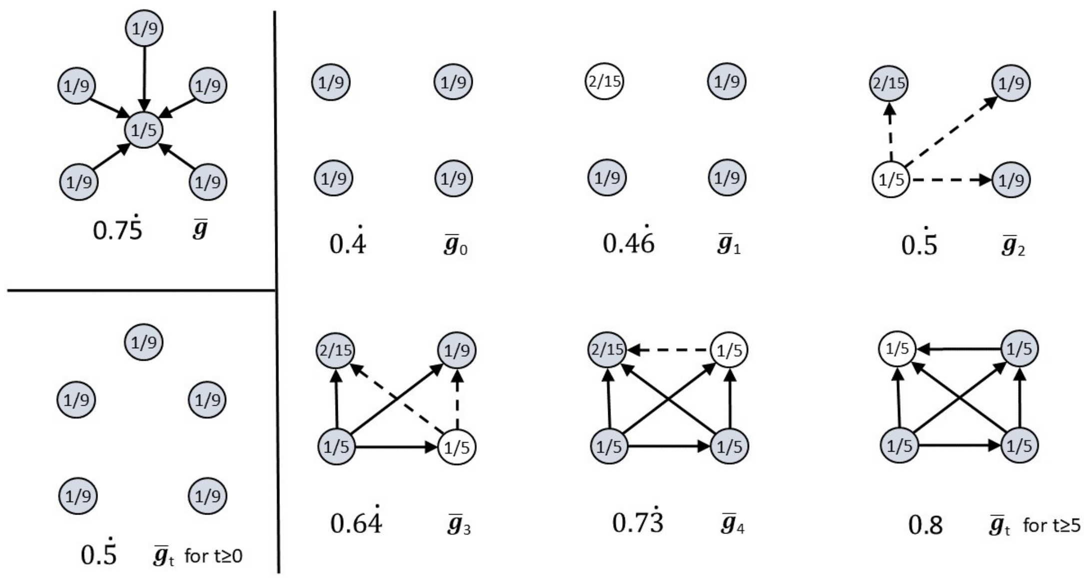
Proposition OA1: *In any NE the undirected network* $\bar{\mathbf{g}}$ *is a nested split graph.*

As in the main part of the paper we assume that the planner aims to minimize the discounted stream of aggregate effort levels and assume that the initial configuration is an equilibrium. Different from the main part of the paper, we now assume the following best response dynamic. At each time period an agent is chosen at random and updates her strategy to a best response to the current configuration $\mathbf{x}_t, \mathbf{g}_t$. We next argue that such a process converges (to an ε equilibrium in finite time). Note first that the function $\Phi(\boldsymbol{x}, \mathbf{g})$, given by

$$\Phi(\boldsymbol{x}, \mathbf{g}) = \alpha \sum_{i\in N} x_i - \frac{1}{2}(\beta+\gamma)\sum_i x_i^2 + \lambda \sum_{i\in N}\sum_{j\in N_i(\bar{\boldsymbol{g}})} x_i x_j - \gamma \sum_{i\in N}\sum_{j\neq i} x_i x_j - \sum_{i\in N}\eta_i(\mathbf{g})\kappa$$

is a potential function for our payoff function. For a vector of parameters $\theta = (\alpha, \beta, \lambda, \gamma)$, we further assume that $2\lambda/(\beta-\gamma) < 1/(n-1)$. This implies that the potential function is bounded. To see this, define $\tilde{\theta}(\theta) = (\tilde{\alpha}(\theta), \tilde{\beta}(\theta), \tilde{\lambda}(\theta), \tilde{\gamma}(\theta))$ with $\tilde{\alpha}(\theta) = \alpha$, $\tilde{\beta}(\theta) = \beta - \gamma$, $\tilde{\lambda}(\theta) = 2\lambda$, $\tilde{\gamma}(\theta) = 2\gamma$. From the first order conditions is then follows that the efficient effort levels, denoted by $\hat{\mathbf{x}}(\bar{\mathbf{g}}, \theta)$, for parameter vector θ are equal to the vector of Nash equilibrium effort levels, $\mathbf{x}(\bar{\mathbf{g}}, \tilde{\theta})$, for parameter vector $\tilde{\theta}$. Since then $\tilde{\alpha}/\tilde{\beta} < 1/(n-1)$ holds, we know from Ballester et al. (2006) that $\mathbf{x}(\bar{\mathbf{g}}, \tilde{\theta})$ exists and is unique. Therefore, for each network $\bar{\mathbf{g}}$ there exists a corresponding maximal and finite value $\Phi(\hat{\mathbf{x}}, \mathbf{g})$ and, since the number of networks $\mathbf{g}$ is finite, $\Phi(\boldsymbol{x}, \mathbf{g})$ is bounded. Note that at each time period t an agent best responds and the potential therefore weakly increases. That is, we obtain an infinite sequence of weakly increasing real numbers that is bounded and therefore converges.

Next we present an example that mirrors the results obtained in Proposition 4 in the main part of the paper. The star network in which peripheral players extend a link to the center is a Nash equilibrium. As before, assume that $e = 2$ and $\gamma = \lambda$. We compare a policy of not intervening with eliminating the center of the star and eliminating the center of the star and a periphery player. Denote the center with n and the periphery player to be eliminated with $n-1$. We assume the same sequence of updating agents, given by $1, 2, ..., n-1$ and $1, 2, ..., n-2$, respectively. Assume this sequence is repeated forever. The initial configuration is depicted on the upper left in the figure below. Note that after eliminating only the central agent, no agent has an incentive to update her strategy, depicted in the lower left of the figure below. Finally, if the center and a peripheral agent is eliminated, then we obtain the sequence on the right. Agents add links until the complete network is reached and the network converges to a Nash equilibrium in $t = 5$. Note that the sum of effort levels in the complete network with 4 agents is strictly larger than in the initial configuration. That is, again eliminating the center and a peripheral player is not only worse than just eliminating the central agent, it is also worse than not intervening at all.



Example OA1: Assume $n = 6$, $\alpha = 1$, $\beta = 1$, $\lambda = 1$, $\gamma = 1$, $\kappa = 9/512$, $e = 2$ and δ is sufficiently large. Effort levels are depicted for each agent and the sum of effort levels below each graph. For the sequence of best responses on the right, each agent chosen to best response is colored white. The initial configuration yields a sum of effort levels of $0.7\dot{5}$. After the elimination of the central agent, the sum of effort levels is $0.\dot{5}$ and agents have no incentives to change their current strategy (this follows from $\lambda = \gamma$). That is, the configuration stays the same in all future periods, as shown in the bottom left of the figure. If, however, the central and a peripheral agent is eliminated, then agents create links, until the process converges to the complete network in $t = 5$. The sum of effort levels is 0.8 from $t = 5$. Therefore, for δ sufficiently large, eliminating the central and a periphery agent is worse than eliminating the central agent or not intervening at all.

9 References

Hiller, T., (2017), Peer Effects in Endogenous Networks, Games and Economic Behavior, Vol. 105, 349-367.

Monderer, D., L. Shapley, (1996), Potential Games, Games and Economic Behavior, Vol. 14, 124-143.