TEXTO PARA DISCUSSÃO

No.705

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December 20, 2024

Abstract

We study the *global* dynamics of the fully *stochastic nonlinear* version of the New Keynesian model and analyze the efficacy of various policies as equilibrium selection tools in this context. First, we unveil a new class of equilibria, characterized by self-fulfilled beliefs about output volatility in recessions, which no conventional Taylor rule can eliminate. An enriched monetary rule specifically targeting risk premia can restore determinacy but becomes infeasible in the presence of a lower bound to interest rates. Second, and in contrast to monetary policy, the fiscal theory of the price level (FTPL) kills all such self-fulfilling volatility. Our main result that FTPL trims volatile equilibria holds in many contexts, including: under an interest rate peg or active Taylor rule, with any degree of price stickiness (including fully rigid prices), with various types of fiscal rules, and with long-term debt.

JEL Codes: E30, E40, G01.

Keywords: New Keynesian, Taylor rules, Fiscal Theory of the Price Level, risk premia, sunspot equilibria, sentiment.

*Khorrami: paymon.khorrami@duke.edu; Mendo: fmendolopez@gmail.com. For helpful feedback, we are grateful to Markus Brunnermeier, Zhengyang Jiang, Dejanir Silva (discussant), Alp Simsek, Quentin Vandeweyer, and participants at the CESifo Macro conference, the Midwest Macro conference, and PUC-Rio. This paper was previously titled "Fear and Volatility at the Zero Lower Bound".

To address topics of inflation, aggregate demand stimulus, and monetary policy, macroeconomists often look to the New Keynesian model for advice. Despite its role as the dominant policy paradigm, this model is plagued by well-known equilibrium multiplicities that influence its answers to those standard macro questions. Currently, there is no consensus on how equilibria are selected and which of the many survive. Among the many alternatives, two popular selection mechanisms are an aggressive monetary policy that responds sufficiently to output and inflation (e.g., the "Taylor principle") versus an active tax and spending policy (e.g., the "Fiscal Theory of the Price Level" or FTPL).

This paper sheds new light on this old controversy by studying the textbook New Keynesian model but with a simple twist: unlike standard practice, we refrain from linearizing the equilibrium around its steady state. Instead, we study the model in its true nonlinear, stochastic form. The model's global dynamics reveal several insights about the nature of the multiplicities and which policies can credibly eliminate them.

As a warm up, and to distinguish our main results that come after, we begin by reviewing the standard deterministic multiplicities in New Keynesian models (Section 2). In that context, we simply generalize the conventional wisdom: an aggressive Taylor rule can eliminate all equilibria except the steady state. The generalizations involve allowing the nonlinear Phillips curve, which does not alter the conclusions at all, and contrasting linear versus nonlinear Taylor rules.

Our main innovations appear when we study stochastic multiplicities (Section 3). We prove the existence, by construction, of a new class of volatile equilibria that no Taylor rule, no matter how aggressive, can completely eliminate. Due to this immunity, our volatile equilibria contrast sharply with the conventional deterministic multiplicities. The key feature that arises in a non-linearized stochastic equilibrium is the presence of a risk premium in agents' Euler equation ("IS curve"). This risk premium, not volatility *per se*, is the source of stochastic equilibrium multiplicities. After examining conventional Taylor rules targeting the output gap and inflation, we proceed to study unconventional Taylor rules that directly target this self-fulfilling risk premium. While sufficiently-strong risk premium targeting can work to restore determinacy, it requires interest rates to become arbitrarily negative. Indeterminacies thus survive these unconventional rules if there is *any* lower bound on policy rates.

Stemming from our volatile equilibria are two additional implications. First, volatility-based indeterminacy is a recessionary phenomenon: it arises only if output is below potential. In a boom, a higher risk premium induces savings that raises agents' consumption growth rate in a way that is unsustainable and thus ruled out as an equilibrium; in a recession, risk premia sustainably push consumption back toward steady state and thus help self-justify the volatility. The intuitive notion that high uncertainty should be paired with below-average demand naturally emerges in our analysis. Second, because the risk premium is a real object, this indeterminacy is a real rather than nominal phenomenon, in contrast to several theories of self-fulfilling inflations or deflations. In particular, all of these results hold even in (but not exclusively in) the rigid-price limit.

We turn next to the FTPL (Section 4), which has been studied mostly in linearized models. We show that, in a variety of different settings—including arbitrary exogenous surplus-to-output ratios, long-term or short-term debt, and some forms of fiscal "rules" that respond to inflation or the output gap—FTPL kills the volatile equilibria we discovered. Why? The basic intuition comes from the basic debt valuation equation:

$$\frac{\text{Nominal debt value}}{\text{Price level}} = \text{Present value of real surpluses} \tag{1}$$

This equation must hold at every point in time. If there is hypothetically any output fluctuation that causes a shock to the present-value of surpluses, this shock must be "absorbed" by either the nominal debt valuation or the price level. Sticky prices say that prices cannot jump arbitrarily, and so the price level cannot absorb such a shock. Can the nominal debt value absorb the shock? In the baseline case with short-term debt, the debt price is fixed at 1, and so the quantity of debt is simply determined by the flow government budget constraint; thus, the nominal value of short-term debt is pinned down and cannot absorb the shock either. In the extended case with long-term debt, the debt price is an additional forward-looking variable that could potentially respond to shocks. But with the additional variable comes an additional constraint, the debt-pricing equation, which puts severe restrictions on how the debt price can move; we show that, in a large class of equilibria, these restrictions are so severe that they can never be consistent with the originally conjectured output shock. Consequently, sunspot demand volatility cannot be self-justified under the FTPL. We conclude that active fiscal policies, in contrast to monetary policies, sharply trim the real indeterminacies endemic to New Keynesian models.

A key takeaway from the discussion above is that FTPL works very differently than the Taylor rule as a selection device. Equation (1) holds at every point in time, essentially steering output volatility at a high frequency. By contrast, an active Taylor rule works by infusing an economy with unstable dynamics, which selects among equilibria by causing all but a subset to explode in the long run.

We think of FTPL's high-frequency steering as "aggregate demand management," because it works by pinning down real demand volatility in all of the settings we study.

One way to see the demand management interpretation transparently is to consider the rigid-price limit; even in this limit without inflation, FTPL succeeds as a selection device. The rigid-price limit case effectively corresponds to inflation-indexed government debt, which the FTPL literature typically regards as ineffectual for equilibrium selection. Nevertheless, demand volatility is pinned down by fiscal considerations. Because FTPL eliminates real sunspot volatility similarly with and without inflation, we advance an interpretation of FTPL as a theory of aggregate demand management, rather than just a theory of the price level.

Related literature. This paper relates to two vast literatures: (i) on New Keynesian indeterminacies and (ii) on FTPL as equilibrium selection. We discuss these and various other connections in sequence.

Going back to Sargent and Wallace (1975), it has been widely recognized that exogenous interest rate paths do not pin down the equilibrium. In related work, Benhabib, Schmitt-Grohé, and Uribe (2001) showed that the zero lower bound (ZLB) can lead to "deflationary trap" equilibria, in which low inflation expectations are self-fulfilled by recessionary deflation. More recently, Benigno and Fornaro (2018) showed there can also be "stagnation trap" equilibria, in which low growth expectations are self-fulfilled by low R&D investment at the ZLB. Set against this background, the present paper describes "volatility trap" equilibria. The main distinctive property of volatility trap equilibria is that risk premia are crucial to the self-fulfilment mechanism.

Volatility trap equilibria are fundamentally nonlinear phenomena. In an important contribution, Caballero and Simsek (2020) study a nonlinear version of the New Keynesian model and illustrate how risk premia are critical to aggregate demand dynamics, but restricting attention to the "fundamental equilibrium." Closely related to our study, contemporaneous work by Lee and Dordal i Carreras (2024) also studies a nonlinear IS curve with risk premia driving the multiplicity. Like us, they also argue that standard "active" Taylor rules do not prune this type of volatility. But the most important difference is our exploration of active fiscal policies as equilibrium selection mechanisms.[1]

There are two important differences between our FTPL analysis and the extant literature.[2] First, we emphasize real indeterminacies and consequently sometimes study the rigid-price limit of the model. This also helps us provide the novel interpretation that

[1]In heterogeneous-agent versions of the New Keynesian model, Acharya and Dogra (2020) and Ravn and Sterk (2021) demonstrate additional scope for indeterminacies when income risk is countercyclical.

[2]Seminal contributions to the FTPL literature include Leeper (1991), Sims (1994), Woodford (1994), Woodford (1995), Kocherlakota and Phelan (1999), and Cochrane (2001). The recent textbook Cochrane (2023) synthesizes many results and presents new ones.

FTPL works via aggregate demand management. Second, we analyze the fully nonlinear, stochastic, global dynamics of the model; despite some technical difficulties involved, we provide formal uniqueness results in several environments. See also Bassetto and Cui (2018), Mehrotra and Sergeyev (2021), Brunnermeier, Merkel, and Sannikov (2023), and Li and Merkel (2020) for fiscal theory applied to a stochastic nonlinear world.[3]

Finally, we unveil novel limitations of Taylor rules as equilibrium selection devices. Following Cochrane (2011)'s results, Neumeyer and Nicolini (2025) have recently shown, in a precise sense, that destabilizing Taylor rules are not credible. Overall, one core message conveyed by our results is that fiscal policies are better suited than monetary policies to trim New Keynesian indeterminacies. As an alternative, future research could investigate the common knowledge perturbation of Angeletos and Lian (2023), which they applied to the linearized New Keynesian model, in a nonlinear setting.

1 Model

We present a canonical New Keynesian economy with complete markets and nominal rigidities. The setup is a continuous-time version of the model exposited in Galí (2015), which the reader can consult for additional details.

Sunspot shocks. Our baseline model features no fundamental uncertainty in preferences or technologies. Nevertheless, we want to allow the possibility that economic objects evolve stochastically due to coordinated behavior. To do this, we introduce a standard Brownian motion Z that is extrinsic to all economic primitives. All random processes will be adapted to Z.

Preferences. The representative agent has rational expectations and time-separable utility with discount rate ρ, unitary EIS, and labor disutility parameter φ:

$$\mathbb{E}\Big[\int_0^\infty e^{-\rho t}\Big(\log(C_t) - \frac{L_t^{1+\varphi}}{1+\varphi}\Big)dt\Big]. \tag{2}$$

Consumption C_t has the nominal price P_t and labor L_t earns the nominal wage W_t.

Technology. The consumption good is produced by a linear technology $Y_t = L_t$. We abstract from fundamental uncertainty (e.g., productivity shocks) for maximal clarity.

[3]Other recent papers studying the FTPL in nonlinear, but deterministic, environments with "liquidity premia" include Berentsen and Waller (2018), Williamson (2018), and Andolfatto and Martin (2018).

Behind the aggregate production function is a structure common to most of the New Keynesian literature. In particular, there are a continuum of firms who produce intermediate goods using labor in a linear technology. These intermediate goods are aggregated by a competitive final goods sector. The elasticity of substitution across intermediate goods is a constant ε. The intermediate-goods firms behave monopolistically competitively and set prices strategically, described next.

Price setting. Intermediate-goods firms set prices strategically, taking into consideration the impact prices have on their demand. Price setting is not frictionless: firms changing their prices are subject to quadratic adjustment costs, a la Rotemberg (1982). (For simplicity, we assume these adjustment costs are non-pecuniary, so that resource constraints are not directly affected by price adjustments.) In the interest of exposition, we relegate the statement of and solution to this standard problem to Appendix B.

Definition: inflation and output gap. Let P_t denote the aggregate price level and $\pi_t := \dot{P}_t/P_t$ its inflation rate. Note also that the flexible-price level of output is given by $Y^* = (\frac{\varepsilon-1}{\varepsilon})^{\frac{1}{1+\varphi}}$. Following the literature, define the output gap $x_t := \log(Y_t/Y^*)$. Conjecture that x_t and π_t have dynamics of the form

$$dx_t = \mu_{x,t}dt + \sigma_{x,t}dZ_t \tag{3}$$

$$d\pi_t = \mu_{\pi,t}dt + \sigma_{\pi,t}dZ_t \tag{4}$$

for some $\mu_x, \sigma_x, \mu_\pi, \sigma_\pi$ to be determined in equilibrium.

Monetary policy. Let ι_t denote the nominal short-term interest rate, which is controlled by the central bank. Monetary policy follows a Taylor rule that targets the output gap and inflation with

$$\iota_t = \bar{\iota} + \Phi(x_t, \pi_t) \tag{MP}$$

for some target rate $\bar{\iota}$ and some response function that satisfies $\Phi(0,0) = 0$. A common linear example that we will use sometimes is

$$\iota_t = \bar{\iota} + \phi_x x_t + \phi_\pi \pi_t. \tag{linear MP}$$

In the main paper, we abstract from the zero lower bound (ZLB), which introduces well-known indeterminacy issues, and analyze it in Appendix F. For now, think of negative interest rates as a proxy for unconventional monetary policy that can work even when the short rate is zero.

Financial markets. Financial markets are complete. Let M_t be the real stochastic discount factor induced by the real interest rate $r_t := \iota_t - \pi_t$ and the equilibrium price of risk h_t associated to the sunspot shock Z_t. The risk-free bond market is in zero net supply—this will be generalized in Section 4 when we introduce fiscal policies. The equity market is a claim on the profits of the intermediate-goods producers. Alternatively, we can think of these profits as being rebated to the consumers lump-sum.

Definition 1. An *equilibrium* is processes $(C_t, Y_t, L_t, W_t, P_t, M_t, B_t, \iota_t, r_t, \pi_t)_{t\geq 0}$, such that

(i) Taking (M_t, W_t, P_t) as given, consumers choose $(C_t, L_t)_{t\geq 0}$ to maximize (2) subject to their lifetime budget and No-Ponzi constraints

$$\frac{B_0}{P_0} + \Pi_0 + \mathbb{E}\Big[\int_0^\infty M_t \frac{W_t L_t}{P_t} dt\Big] \geq \mathbb{E}\Big[\int_0^\infty M_t C_t dt\Big] \tag{5}$$

$$\lim_{T\to\infty} M_T \frac{B_T}{P_T} \geq 0, \tag{6}$$

where Π represents the real present-value of producer profits and B represents the bond-holdings of the consumer.[4]

(ii) Firms set prices optimally, subject to their quadratic adjustment costs.

(iii) Markets clear, namely $C_t = Y_t = L_t$ and $B_t = 0$.

(iv) The central bank follows the interest rate rule (MP) for some target rate $\bar{\iota}$ and some response function $\Phi(\cdot)$.

In what follows, we refer to a *deterministic equilibrium* as an equilibrium with no real volatility, $\sigma_x \equiv 0$. A *sunspot equilibrium* is an equilibrium with real volatility, $\sigma_x \neq 0$.

Equilibrium characterization. We first provide a summary characterization of all equilibria. Labor supply and consumption decisions satisfy the following optimality conditions:

$$e^{-\rho t} L_t^{\varphi} = \lambda M_t \frac{W_t}{P_t} \tag{7}$$

$$e^{-\rho t} C_t^{-1} = \lambda M_t, \tag{8}$$

[4] In addition, to prevent arbitrages like "doubling strategies" that can arise in continuous time, we must impose a uniform lower bound on borrowing, e.g., $M_t B_t / P_t \geq -\underline{b}$, although $\underline{b}$ can be arbitrarily large.

where λ is the Lagrange multiplier on the lifetime budget constraint (5).

On the firm side, Appendix B shows that optimal firm price setting gives rise to aggregate inflation dynamics that satisfy

$$\mu_{\pi,t} = \rho\pi_t - \eta\varepsilon\frac{W_t}{P_t} + \eta(\varepsilon - 1), \tag{9}$$

where η is each firm's degree of price flexibility. Notice that as $\eta \to 0$ (prices changes become infinitely costly), one possible equilibrium is to have $\pi_t \to 0$ for all times. We will assume this "rigid-price limit" is the equilibrium that obtains as $\eta \to 0$.

We use these conditions to obtain an "IS curve" and a "Phillips curve." Applying Itô's formula to (8), we obtain the consumption Euler equation, which may be rewritten in terms of the output gap as

$$\mu_{x,t} = \iota_t - \pi_t - \rho + \frac{1}{2}\sigma_{x,t}^2. \tag{IS}$$

Equation (IS) is the IS curve. Next, divide the FOCs (7)-(8), and use goods and labor market clearing $C_t = Y_t = L_t$ to get $Y_t^{1+\varphi} = \frac{W_t}{P_t}$. Substitute this expression into (9) to obtain

$$\mu_{\pi,t} = \rho\pi_t - \kappa\Big(\frac{e^{(1+\varphi)x_t} - 1}{1+\varphi}\Big), \tag{PC}$$

where $\kappa := \eta(\varepsilon - 1)(1+\varphi)$. Equation (PC) is the Phillips curve.

Together with the monetary policy rule (MP), equations (IS) and (PC) form the non-linear "three equation model" in standard New Keynesian models. An equilibrium is completely characterized by these three equations, along with conditions that ensure that any output gap or inflation explosions are consistent with optimization behavior. For example, since $C_t = e^{x_t}Y^* = L_t$, the representative agent would obtain minus infinite utility if $x_t = \pm\infty$ in finite time, or even if $x_t \to \pm\infty$ too quickly. Clearly, this is not compatible with optimizing behavior if the agent has an alternative that delivers finite utility. Consumers would be individually better off ignoring signals to coordinate, unravelling such a proposed allocation. (A straightforward example is the case when $x_t \to \infty$, since this implies that real wages are diverging towards plus infinity, and agents may obtain finite utility simply by working a finite amount forever.) Similarly, we show in Appendix B that firms' optimization rules out situations when $\pi_t \to \pm\infty$ too quickly, because that would induce an infinite amount of price adjustment costs.

In order to emphasize that the multiplicity unveiled later does not rely on the explosive behavior and to streamline the analysis, we only consider equilibria that satisfy a simple condition that conforms with the existing literature and rules out both finite-time and asymptotic explosions:

Condition 1. *A non-explosive allocation has* $\mathbb{E}|x_t| < \infty$ *and* $\mathbb{E}|\pi_t| < \infty$ *for almost all* t*, and*

$$\limsup_{T\to\infty} \mathbb{E}|x_T| < \infty \quad \textit{and} \quad \limsup_{T\to\infty} \mathbb{E}|\pi_T| < \infty. \tag{10}$$

We summarize our characterization in the following lemma.

Lemma 1. *Suppose processes* $(x_t, \pi_t, \iota_t)_{t\geq 0}$ *satisfy (IS), (PC), (MP), and Condition 1. Then,* $(x_t, \pi_t, \iota_t)_{t\geq 0}$ *corresponds to an equilibrium of Definition 1.*

The proof of this lemma is standard except for a careful treatment of potentially explosive behavior. Condition 1 not only ensures that utility for the representative consumer and firm are finite but also, together with the other equations in Lemma 1, is sufficient to verify that their transversality conditions hold. See Appendix A.1 for these arguments. Going forward, we will want to make reference only to equilibria which satisfy Condition 1. For that reason, we include the following definition.

Definition 2. A *non-explosive equilibrium* is an equilibrium of Definition 1 in which Condition 1 holds.

Linearized Phillips curve approximation. We will occasionally use a linearized Phillips curve in place of (PC). Since $e^{(1+\varphi)x_t} - 1 \approx (1+\varphi)x_t$, the Phillips curve to first order is

$$\mu_{\pi,t} = \rho\pi_t - \kappa x_t. \tag{linear PC}$$

We will occasionally work with (linear PC) instead of (PC), because as will become clear the nonlinearity in (IS) is the critically novel element, and not so much the nonlinearity in (PC). (We do some analysis with the nonlinear Phillips curve in Appendix C.) In this approximation, we will sometimes refer to non-explosive equilibrium as $(x_t, \pi_t, \iota_t)_{t\geq 0}$ that satisfy (IS), (linear PC), (MP), and Condition 1.

2 Deterministic Equilibria

We start by describing equilibria without volatility, $\sigma_x \equiv 0$. We illustrate the basic indeterminacy that arises in New Keynesian models and how aggressive monetary policy rules can eliminate this indeterminacy. In the appendix, we generalize these existing results to nonlinear Phillips curves and nonlinear Taylor rules as well.

2.1 Review: conventional indeterminacy and the Taylor principle

There is always an equilibrium with $x = \pi = 0$ forever. Using (IS), this equilibrium is supported by a monetary policy rule with $\bar{\imath} = \rho$.

Can there exist other equilibria? As is well known, the answer to this question hinges on the stability/instability properties of the equilibrium dynamical system for (x_t, π_t). We will review this analysis here. First, we specialize to the policy rule (linear MP) with the target rate $\bar{\imath} = \rho$. Combining (linear MP) with (IS), the dynamics of x_t are given by

$$\dot{x}_t = \phi_x x_t + (\phi_\pi - 1)\pi_t. \tag{11}$$

The IS curve is linear in a deterministic equilibrium with a linear Taylor rule. We also consider here the linear Philips curve (linear PC) as in most of the existing literature.

The typical determinacy analysis picks an aggressive Taylor rule that renders the above system unstable. The system (11) and (linear PC) can be written in matrix form as

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t \end{bmatrix} = \mathcal{A} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix}, \quad \text{where} \quad \mathcal{A} := \begin{bmatrix} \phi_x & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix}. \tag{12}$$

The eigenvalues of $\mathcal{A}$ are both strictly positive, and the system unstable, if $\phi_x > -\rho$ and $\phi_\pi > 1 - \rho\phi_x/\kappa$. This is the continuous-time version of the eigenvalue conditions in Blanchard and Kahn (1980).

By contrast, if either parameter condition is violated, then the system has one or two stable eigenvalues. In such case, there are a continuum of non-explosive equilibria, which one can index by the initial conditions (x_0, π_0). As the explicit parameter conditions make clear, instability occurs when monetary policy is sufficiently aggressive (i.e., "active") in responding to the output gap and inflation, whereas stability occurs when the response function is less aggressive (i.e., "passive"). The proof of the following proposition and all subsequent theoretical results in Sections 2-3 is contained in Appendix A.2.

Proposition 1. *Consider the linearized Phillips curve (linear PC) and monetary policy rule (linear MP) with $\bar{\imath} = \rho$. If $\phi_x > -\rho$, and $\phi_\pi > 1 - \rho\phi_x/\kappa$, the only non-explosive equilibrium is $(x_t, \pi_t) = (0, 0)$ forever. If either condition is violated, then a continuum of linear non-explosive equilibria exist.*

Remark 1 (Nonlinear Phillips curve)**.** *We have used the linearized Phillips curve here for simplicity and exposition. We analyze the nonlinear Phillips curve in Appendix C, and the conclusion is similar to Proposition 1 but the proof is more complicated.*

Remark 2 (Explosive equilibria)**.** *A key clause is the requirement that equilibria satisfy Condition 1, ruling out explosions. What if asymptotic explosions were permitted, while finite-time explosions were ruled out? It turns out that, by adopting an aggressive nonlinear Taylor rule, monetary policy can force an explosion in finite-time, and hence select a unique equilibrium. We analyze this situation in Appendix D. In that sense, the spirit of Proposition 1 is preserved even under broader notions of equilibrium.*

3 A New Class of Sunspot Equilibria

Now, we demonstrate several new results pertaining to volatility in New Keynesian models. The deterministic multiplicity in Proposition 1 already suggests the existence of stochastic sunspot equilibria, but only if monetary policy is insufficiently aggressive. In this section, we will show that this logic is wrong, in particular because of the presence of risk premia. As we will then show, a different type of policy rule, which targets the risk premium, is required to eliminate stochastic multiplicities. Finally, we argue that risk premium targeting is fragile because it requires arbitrarily negative interest rates.

3.1 Constructing volatile equilibria

For concreteness, assume that prices are permanently rigid, i.e., $\kappa \to 0$. This clarifies that we are focusing on real indeterminacy rather than inflation indeterminacy. An additional advantage is that we only need to study the dynamics of the output gap, rather than a two-dimensional stochastic system. Start with an example policy rule with target rate $\bar{\imath} = \rho$ and nonlinear response function

$$\Phi(x, \pi) = \phi_x(e^x - e^{-x}), \quad \phi_x \geq 0. \tag{13}$$

Rule (13) brings theoretical clarity to the discussion. This is a super-aggressive policy rule, evidently more aggressive than its linear approximation $2\phi_x x$. It would send the deterministic economy to a finite-time explosion unless $x_t = 0$ forever (see Appendix D), thus selecting a unique deterministic equilibrium.

Nevertheless, stochastic indeterminacy exists. Combining (MP) with (13) and (IS), the drift of x_t is given by

$$\mu_x = \phi_x(e^x - e^{-x}) + \frac{1}{2}\sigma_x^2.$$

Building off of the previous analysis, the question is whether the dynamical system characterized by (μ_x, σ_x) keeps x_t finite forever. But here, the volatility σ_x is determined purely via coordination, and some choices will lead to stability. To see why this is possible, examine instead the dynamics of the level output gap $y_t := e^{x_t}$ and verify that $0 < y_t < \infty$ forever (and furthermore that a non-degenerate stationary distribution exists). By Itô's formula, the drift and diffusion of y_t are

$$\mu_y = \phi_x(y^2 - 1) + y\sigma_x^2 \quad \text{and} \quad \sigma_y = y\sigma_x$$

Right away, we see that stability is possible, if agents coordinate on sufficiently high volatility. For example, suppose for some constant $\nu > 0$,

$$\sigma_x^2 = \begin{cases} \left(\frac{\nu}{y}\right)^2 + \phi_x \frac{1-y^2}{y}, & \text{if } y < 1; \\ 0, & \text{if } y \geq 1. \end{cases} \tag{14}$$

Putting these equations together, the dynamics for y_t would be

$$dy_t = \begin{cases} \frac{\nu^2}{y_t}dt + \sqrt{\nu^2 + \phi_x y_t(1 - y_t^2)}dZ_t, & \text{if } y_t < 1; \\ \phi_x(y_t^2 - 1)dt & \text{if } y_t \geq 1. \end{cases} \tag{15}$$

It is relatively intuitive to see that $y_t > 0$ for all t in this example: if y ever approached 0, the drift $\frac{\nu^2}{y}$ would explode fast enough to push y back up. Formalizing this mathematically, the process in (15) behaves asymptotically (as $y \to 0$) like a Bessel(3) process, which never hits 0 (more precisely, 0 is an "entrance boundary" for this process). And consequently, $x_t = \log(y_t)$ does not explode negatively.[5] Provided $y_0 \leq 1$, the process also does not explode positively. In fact, because there is no volatility for $y_t \geq 1$, the process will eventually converge to and stay stuck at the efficient level $y_t = 1$ (i.e., the sunspot volatility is only temporary in this example). This entire construction works for any $\nu > 0$, so think of ν as a parameter indexing possible sunspot equilibria. In summary, despite the super-aggressive response function (13), many equilibria exist with

[5] A Bessel(3) process corresponds to the solution of $dX_t = X_t^{-1}dt + dZ_t$ where dZ_t is a one-dimensional Brownian motion. A Bessel(3) process is also equivalent to the Euclidean norm of a three-dimensional Brownian motion and therefore satisfies $X_t > 0$ for all $t > 0$. Taking the limit of dy_t as $y \to 0$, we can see that $\nu^{-1}y$ behaves exactly as a Bessel(3) process.

different σ_x.

As mentioned, the particular construction above only features transitory volatility. That was only to develop an initial understanding and is easily generalized. For example, suppose agents coordinate on the following volatility process for some $\delta \in (0,1)$:

$$\sigma_x^2 = \begin{cases} \left(\frac{\nu}{y}\right)^2 + \phi_x \frac{1-y^2}{y}, & \text{if } y < 1-\delta; \\ 0, & \text{if } y \geq 1-\delta. \end{cases} \tag{16}$$

The induced dynamics of $y_t = e^{x_t}$ are

$$dy_t = \begin{cases} \frac{\nu^2}{y_t} dt + \sqrt{\nu^2 + \phi_x y_t (1-y_t^2)} dZ_t, & \text{if } y_t < 1-\delta \\ \phi_x (y_t^2 - 1) dt & \text{if } y_t \geq 1-\delta. \end{cases} \tag{17}$$

Provided $y_0 < 1$, this process will eventually exit the deterministic region, enter the volatile region, and remain inefficiently volatile for an infinite amount of time.[6] Figure 1 presents a numerical construction of this example, showing that the economy is permanently inefficient ($y < 1$), volatility is not transitory, and nevertheless Condition 1 holds and there exists a stationary distribution for $y = e^x$.

The key reason for equilibrium multiplicity is the risk premium, not volatility per se. To see this clearly, contrast the *linearized* version of the Euler equation, which without a risk premium is

$$\mu_x = \iota - \pi - \rho.$$

Repeating the above analysis in this linearized world, (15) would be replaced by

$$dy_t = \begin{cases} \phi_x (y_t^2 - 1) dt + \sqrt{\nu^2 + \phi_x y_t (1-y_t^2)} dZ_t, & \text{if } y_t < 1; \\ \phi_x (y_t^2 - 1) dt, & \text{if } y_t \geq 1. \end{cases} \tag{18}$$

The process in (18) behaves like an arithmetic Brownian motion with negative drift for $y_t \approx 0$. Consequently, one would conclude from the linearized model that $y_t \to 0$ in finite time with positive probability—in violation of Condition 1. Thus, the only possible linearized non-explosive equilibrium can be $y_t = 1$ at all times. A very aggressive Taylor rule trims equilibria in this linearized stochastic world, exactly as in the deterministic

[6]To see all these points, note that the process has zero volatility and negative drift when $y \in (1-\delta, 1)$; therefore, the process exits the region $(1-\delta, 1)$ and enters $(0, 1-\delta)$ in finite time almost-surely. Upon entering the volatile region $(0, 1-\delta)$, the process can move around but will never reach $y = 0$, by the same Bessel(3) argument established in the text. Finally, the stationary distribution will additionally have a mass point at $y = 1-\delta$, because the dynamics induce y_t to visit the point $1-\delta$ infinitely often.

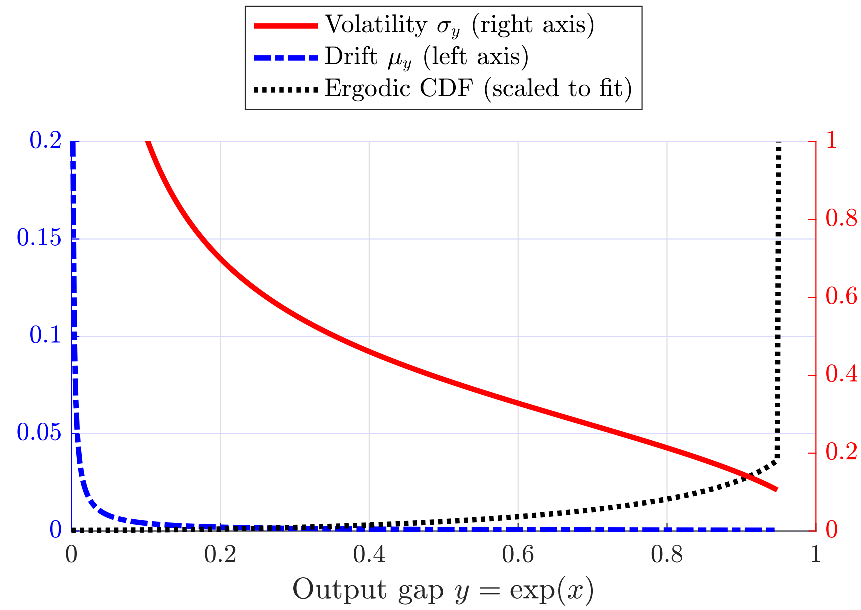

Figure 1: Equilibrium with rigid prices ($\kappa \to 0$) and dynamics given by equations (16)-(17). The stationary CDF is computed via a discretized Kolmogorov Forward equation. The resulting stationary CDF features a mass point at $y = 1 - \delta$. Parameters: $\rho = 0.02$, $\nu = 0.02$, $\delta = 0.05$, $\phi_x = 0.1$.

equilibria. It is easy to verify that a similar analysis applies for arbitrary choices of σ_x.

The analysis so far is confined to a particular example with a specific monetary policy. But perhaps monetary policy could act even more aggressively and eliminate the risk premium effect. Is there some Taylor rule that can kill these equilibria? No. Agents can always coordinate on a level of volatility that keeps $y_t > 0$ (equivalently, $x_t > -\infty$) for *any* level of aggression in the Taylor rule satisfying the following mild regularity condition:[7]

Condition 2. *There exists $\beta > 0$ such that $\Phi(x)$ satisfies $\lim_{x \to -\infty} e^{\beta x} \Phi(x) > -\infty$.*

Proposition 2. *Suppose prices are rigid ($\kappa \to 0$). Consider any Taylor rule (MP) with $\bar{\imath} = \rho$ that is increasing in x and satisfies Condition 2. Then, there exist a continuum of non-explosive sunspot equilibria indexed by $x_0 < 0$ and the volatility function $\sigma_x(x)$. The volatility function can be any mapping $\sigma_x : \mathbb{R} \mapsto \mathbb{R}$ that is finite for all $x \in (-\infty, 0)$ and satisfies suitable boundary conditions as $x \to -\infty$ and $x \to 0$.*

Intuitively, the idea behind Proposition 2 is contained in the example construction above. For any Taylor rule, agents can coordinate on a level of volatility that "undoes" the effect of interest rates on output gap dynamics. The central bank tries to destabilize

[7]Note that any rule that depends on output gap in a exponential or polynomial fashion, which include all rules considered in this paper, satisfy this condition.

the economy, and agents coordinate on a risk premium that stabilizes it. We also emphasize a technical point regarding the fact that $\sigma_x(x)$ can essentially be any function satisfying suitable "boundary conditions": when one cares about global stability as we do, all that matters are boundary conditions on the equilibrium dynamics, rather than a local analysis around the fundamental equilibrium $(x, \pi) = (0, 0)$.

The construction above also suggests that self-fulfilling volatility is *recessionary*. Risk premia σ_x^2 always provide a positive force that increases the drift μ_x and buoys the output gap. In a recession (i.e., $x < 0$), this stabilizes the economy, pushing x back toward zero, and provides the needed dynamic self-justification. But in a boom (i.e., $x > 0$), risk premia would send the economy further and further away from steady state, which is destabilizing. Self-fulfilling volatility is thus generally recessionary.

Proposition 3. *Suppose prices are rigid ($\kappa \to 0$). There exist Taylor rules (MP) such that (i) volatile non-explosive equilibria exist and (ii) all non-explosive equilibria have $x_t \leq 0$ forever.*

Remark 3. *For analytical convenience, Propositions 2-3 are proved in the rigid price limit. However, the same intuition carries over to a world with partially-flexible prices. Indeed, Proposition E.1 in Appendix E constructs a similar recessionary sunspot equilibrium in which both inflation and the output gap are stochastic.*

Comparing Proposition 1 to Propositions 2-3, our analysis sharply distinguishes stochastic sunspot equilibria from their deterministic equilibria. Typically, they are tightly linked: one often constructs the sunspot equilibria as a "lottery" over deterministic equilibria (Azariadis, 1981). Here instead, the presence of risk premia means that stochastic equilibria can have a markedly different character than their deterministic counterparts: they work through the risk premium and are necessarily recessionary.

This result that stability properties can flip in the nonlinear stochastic model is in contrast to the conventional wisdom regarding such models. For example, Cochrane (2023) writes

> this book is really about the broad determinacy and stability properties of monetary models. In one sense, the conclusions of these simple models are likely to be robust, because stability and determinacy depend on which eigenvalues are greater or less than 1. As long as a model modification does not move an eigenvalue across that boundary, the stability and determinacy conclusions are not changed. (Chapter 5.8)

While stability properties may be of some theoretical interest, a practical question we ask next is whether some policies can help trim or eliminate such sunspot equilibria.

3.2 Risk premium targeting

There is one type of rule that can restore determinacy. Following the suggestion in Lee and Dordal i Carreras (2024), suppose we replace the plain-vanilla Taylor rule (MP) with a rule that explicitly targets the risk premium. However, we will provide a much broader proposition regarding the efficacy of this rule. We use

$$\iota_t = \rho + \Phi(x_t, \pi_t) - (\alpha_- \mathbf{1}_{\{x_t<0\}} + \alpha_+ \mathbf{1}_{\{x_t>0\}})\sigma_{x,t}^2. \tag{MP-vol}$$

Although conventional wisdom would suggest that targeting an asset price—which maps one-to-one into the output gap—suffices to target the risk premium, that is not true here, intuitively because coordination on a fearful equilibrium can raise uncertainty $\sigma_{x,t}$ independently, i.e., without affecting x_t in the short run. Rule (MP-vol) directly targets the uncertainty that generates risk premia.

To see how risk premium targeting restores determinacy, substitute rule (MP-vol) into (IS) and rewrite the resulting dynamics in terms of the level output gap $y_t = e^{x_t}$:

$$dy_t = y_t\Big[\Phi(x_t, \pi_t) - \pi_t + (1 - \alpha(x_t))\sigma_{x,t}^2\Big]dt + y_t \sigma_{x,t} dZ_t, \tag{19}$$

and where $\alpha(x) := \alpha_- \mathbf{1}_{\{x<0\}} + \alpha_+ \mathbf{1}_{\{x>0\}}$ is the state-dependent responsiveness to the risk premium. If $\alpha_- = \alpha_+ = 1$, then the risk premium vanishes from the drift, and we are back in a situation where an aggressive response function Φ can trim equilibria by destabilizing the economy. If $\alpha_+ < 1 < \alpha_-$, then the risk premium itself becomes destabilizing: higher levels of $\sigma_{x,t}^2$ make the drift push x_t further away from zero. Therefore, a modified Taylor rule like (MP-vol), with more aggressive risk premium targeting in bad times, can always eliminate equilibrium multiplicity. Again, for analytical purposes, we state this result in the rigid price limit, with the proof in Appendix A.

Proposition 4. *Suppose prices are rigid ($\kappa \to 0$). With sufficiently strong risk premium targeting ($\alpha_+ \leq 1 \leq \alpha_-$) and sufficiently aggressive responsiveness to the output gap, the modified Taylor rules (MP-vol) ensure that, among equilibria where $e^{-x}\sigma_x(-x)$ and $e^{-x}\sigma_x(x)$ remain bounded as $x \to \infty$, the unique non-explosive equilibrium is $x_t = 0$.*

The deep difference between the multiplicity of sunspot equilibria and the multiplicity of deterministic equilibria was the presence of a stabilizing risk premium. And this manifests in a qualitatively distinct policy response to restore determinacy: by targeting the risk premium, with the interest rate moving more than one-for-one in bad times, the central bank can use it as a destabilizing nuclear threat.

Remark 4 (Partially-flexible prices)**.** *For analytical convenience, Proposition 4 is proved in the rigid price limit. A similar type of analysis could be done with partially-flexible prices, but it is much more tedious and technical. Here is a sketch of the idea. Aggressive risk premium targeting ($\alpha_- \geq 1$) reduces the drift of x_t, so we have that $x_t \leq \tilde{x}_t$, where $\tilde{x}_t$ follows a related process with $\alpha_- = 1$:*

$$d\tilde{x}_t = \Big[\Phi(\tilde{x}_t, \pi_t) - \pi_t - \frac{1}{2}\sigma_x(\tilde{x}_t)^2\Big]dt + \sigma_x(\tilde{x}_t)dZ_t, \quad \tilde{x}_0 = x_0.$$

As long as $\tilde{x}_t \to -\infty$ in finite time, so does x_t. But the analysis of $\tilde{x}_t$ is already covered for a relevant class of Taylor rules. For example, imagine the central bank chooses $\Phi(x, \pi) = \phi_x(e^x - e^{-x}) + \pi$. Then, the dynamics of $\tilde{y}_t = e^{\tilde{x}_t}$ are

$$d\tilde{y}_t = \phi_x(\tilde{y}_t^2 - 1)dt + \tilde{y}_t\sigma_x(\log(\tilde{y}_t))dZ_t.$$

Among equilibria in which the function $\tilde{y} \mapsto \tilde{y}\sigma_x(\log(\tilde{y}))$ is well-behaved, one can prove that $\tilde{y}_t$ hits zero in finite time with positive probability (because it behaves like an arithmetic Brownian motion with negative drift as $\tilde{y}_t \approx 0$). This shows that $\tilde{x}_t \to -\infty$ hence $x_t \to -\infty$ in finite time with positive probability. Using the technique in Appendix D, this argument can then be extended to a response function Φ featuring greater than one-to-one response to inflation.

3.3 Feasibility of aggressive Taylor rules

The policy rules suggested by our analysis, while theoretically interesting, are very aggressive. Are such extreme rules credible?

An immediate thought, following Cochrane (2011) and formalized in Neumeyer and Nicolini (2025), is that "blow up the world" nuclear threats are generically not credible. If the economy ever followed a path away from $x = 0$, could policymakers really commit to sending $x \to -\infty$?

Another peculiarity is that all the rules advocated above share the property that $\iota_t \to -\infty$ as $x_t \to -\infty$. This was necessary, in fact: any policy where ι_t remains bounded from below by $\iota_t \geq \underline{\iota}$ cannot trim equilibria. To see this, consider the rigid-price equilibria and inspect output gap dynamics when ι_t is at its lower bound:

$$dx_t = \Big[\underline{\iota} - \rho + \frac{1}{2}\sigma_{x,t}^2\Big]dt + \sigma_{x,t}dZ_t, \quad \text{when} \quad x_t < 0. \tag{20}$$

In the stochastic case, a sufficiently high level of uncertainty can raise the drift and create stable stochastic dynamics. For instance, suppose agents coordinate on a constant variance $\sigma_x^2 > 2(\rho - \underline{\iota})$ when $x_t < 0$ (and zero when $x_t \geq 0$). This induces x_t to behave like an arithmetic Brownian motion with positive drift at the lower bound $\iota = \underline{\iota}$. By the well-known fact that a positive-drift arithmetic Brownian motion has $+\infty$ as its limit, we establish that $\liminf_{t\to\infty} \mathbb{E}[x_t \mathbf{1}_{x_t<0}] > -\infty$ almost-surely, so Condition 1 is satisfied. We have just proved the following.

Proposition 5. *Suppose prices are rigid ($\kappa \to 0$). If interest rates are lower bounded, $\iota_t \geq \underline{\iota}$, then any $x_0 \leq 0$ corresponds to at least one valid non-explosive equilibrium.*

With a zero lower bound (ZLB), or *any* lower bound, certain monetary threats are not credible. The reason is precisely that our sunspot equilibria are *recessionary and self-sustained by high risk premia*. In particular, suppose we are in a hypothetical recession (i.e., low x). An active monetary policy seeking to eliminate this hypothetical recession would want to set interest rates very low, thereby impounding a very negative drift to consumption growth. But the lower bound $\iota_t \geq \underline{\iota}$ prevents such a force from being too strong. In that case, risk premia can be so high as to stabilize the economy, outweighing the destabilizing effect of monetary policy.

We further analyze the ZLB case in Appendix F. There, we even generalize policy by allowing for *optimal discretionary monetary policy*, following the work of Caballero and Simsek (2020), and yet a tremendous amount of equilibrium multiplicity remains, precisely because policy is constrained at the ZLB. The mechanics at play are well-described as a "volatility trap": volatility rises, pushes the economy to the ZLB, and then keeps it trapped there, because of the stabilizing effect of risk premia.

In a world with rate constraints such as the ZLB, what restores determinacy? One possibility we pursue next is the Fiscal Theory of the Price Level (FTPL).

4 Fiscal Theory

Let us now explore a version of "Fiscal Theory of the Price Level" (FTPL). The idea here is to propose some fiscal policies that can prune equilibria. Our contribution to the literature is analysis of FTPL in a nonlinear stochastic monetary model.

We formulate fiscal policy in a particularly transparent situation: lump-sum taxation with government transfers to the representative household. Denote the lump-sum taxes levied by τ_t and the transfers by ξ_t, both in real terms. The real primary surplus of the government is then

$$S_t := \tau_t - \xi_t.$$

Since the government can pick both taxes and transfers, it can effectively choose S_t.

Taxes and transfers do not necessarily offset, so the government borrows by issuing short-term nominally riskless bonds B_t. Later we will generalize to long-term debt. The flow budget constraint of the government is

$$\dot{B}_t = \iota_t B_t - P_t S_t. \tag{21}$$

The nominal interest rate ι_t will be controlled by monetary policy.

Because of the lump-sum nature of the taxes and transfers, there is no impact on the household optimality conditions. Essentially, Ricardian equivalence holds. Indeed, the present-value formula for government debt is

$$\frac{B_t}{P_t} = \mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t} S_u du\Big], \tag{GD}$$

where M denotes the real stochastic discount factor process (this is because the transversality condition $\lim_{T\to\infty} \mathbb{E}_t[M_T B_T/P_T] = 0$ holds in our representative agent setup). While the representative household holds the government bonds B_t, it also owes the government future taxes and is owed future transfers. Therefore, the lifetime budget constraint of the representative household is

$$\mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t}\frac{W_u L_u}{P_u} du\Big] + \frac{B_t}{P_t} = \mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t} S_u du\Big] + \mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t} C_u du\Big].$$

By (GD), the lifetime budget constraint is equivalent to the budget constraint without any debt at all. And so the household consumption FOC is unchanged.

For reference, let us restate the IS curve (IS) and Phillips curve (PC) as the following dynamical system in terms of (x_t, π_t):

$$dx_t = \Big[\iota_t - \pi_t - \rho + \frac{1}{2}\sigma_{x,t}^2\Big]dt + \sigma_{x,t} dZ_t \tag{22}$$

$$d\pi_t = \Big[\rho\pi_t - \kappa\Big(\frac{e^{(1+\varphi)x_t} - 1}{1+\varphi}\Big)\Big]dt + \sigma_{\pi,t} dZ_t. \tag{23}$$

Together with some nominal interest rate rule for ι_t and some surplus rule for S_t, equilibrium is fully characterized by the government debt valuation (GD) and the dynamical system (22)-(23). We continue to require the non-explosion Condition 1.

Our previous results did not have government debt or taxes/transfers. However, everything we have said until now still holds with fiscal policies, so long as those policies are "passive" in the language of Leeper (1991). In particular, suppose fiscal policies are chosen so that equation (GD) always holds. Then, government debt valuation plays no role in the analysis, and by the Ricardian equivalence property shown above, the equilibria must be identical to those in Sections 2-3. Next, we explore what happens when fiscal policies are "active," as opposed to passive.

4.1 FTPL as equilibrium selection: the key argument

Consider, as a first example, a fiscal policy with real primary surpluses given by

$$S_t = \bar{s} Y_t, \quad \text{with} \quad \bar{s} > 0. \tag{24}$$

This policy is "active" because its real levels are chosen in a way that does not automatically ensure the government debt valuation equation holds (e.g., S_t is independent of the price level). Such proportional surpluses are also quite natural, in that they arise in the real world the case of proportional taxes and transfers—although we abstract from the distortionary effects of such policies.

With this policy, and using (GD) along with the consumption FOC (8), we have

$$\frac{B_t}{P_t} = \bar{s}\mathbb{E}_t \int_t^\infty e^{-\rho(u-t)} \frac{Y_t}{Y_u} Y_u du = \rho^{-1} \bar{s} e^{x_t} Y^*. \tag{25}$$

Now, apply Itô's formula to (25), using the fact that $dP_t/P_t = \pi_t dt$, to get

$$\Big[\frac{B_t}{P_t} \iota_t - \bar{s} Y_t - \frac{B_t}{P_t} \pi_t \Big] dt = \rho^{-1} \bar{s} e^{x_t} Y^* [\iota_t - \pi_t - \rho + \sigma_{x,t}^2] dt + \rho^{-1} \bar{s} e^{x_t} Y^* \sigma_{x,t} dZ_t.$$

Matching the "dZ" terms on both sides, we find $\sigma_{x,t} = 0$. Then, matching the "dt" terms on both sides, we find an identity: given $\sigma_{x,t} = 0$, and using equation (25), the "dt" terms match for any ι_t, π_t, and x_t. In other words, the FTPL selects $\sigma_{x,t} = 0$, and that is all it does after the initial date $t = 0$. This argument is completely independent of the level of price stickiness κ and amount of self-fulfilling inflation volatility $\sigma_{\pi,t}$ (if any).

It turns out this same logic holds even if the surplus-to-output ratio is not constant but *almost any* exogenous process. In particular, let Ω_t be an exogenous vector Markov diffusion, driven by a multivariate Brownian motion $\mathcal{Z}$ that is independent of the sunspot shock Z. Ω is the state vector describing fiscal policy. Let $s_t = s(\Omega_t)$ for some function $s(\cdot)$, and suppose

$$S_t = s(\Omega_t) Y_t. \tag{26}$$

Of course, allowing surplus shocks through Ω_t does alter the IS curve, which we take into account in the appendix. Even in this more general specification, the following theorem holds. The proof of all results in this section can be found in Appendix A.4, with some important preliminary characterizations provided in Appendix A.3.

Theorem 1. *The economy with fiscal policy following (26) necessarily has $\sigma_{x,t} = 0$. Conversely, if $\sigma_{x,t} = 0$, and if dx_t takes a particular loading on the surplus shocks $d\mathcal{Z}_t$, then the government debt valuation equation (GD) automatically holds at every date, given it holds at $t = 0$.*

Theorem 1 says that FTPL (i) *selects equilibria without real self-fulfilling volatility* and (ii) *does nothing else besides pin down real demand shocks*. This is surprising because we usually think of FTPL as selecting inflation or the price level. We elaborate on (i)-(ii) in turn.

Why FTPL eliminates volatility. The mathematical reasoning for why FTPL selects these equilibria is quite simple in this case: the aggregate real debt balance B_t/P_t evolves "locally deterministically" (meaning it only has drift and no diffusion over small time intervals dt), and so its present value $\mathbb{E}_t[\int_t^\infty \frac{M_u}{M_t} S_u du]$ must also not have any diffusion. This then implies x_t must not have any sunspot volatility. Indeed, in all the surplus specifications considered so far, $S_t/Y_t = s_t = s(\Omega_t)$ is an exogenous process. And so you can break up the real present value of surpluses into two components:

$$\mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t} S_u du\Big] = Y_t \times \underbrace{\mathbb{E}_t\Big[\int_t^\infty e^{-\rho(u-t)} s_u du\Big]}_{\text{exogenous for now}}, \tag{27}$$

where we have used the consumption FOC (8) to replace the SDF with $M_u = e^{-\rho u} Y_u^{-1}$. The second term is exogenous and only driven by the surplus shocks $d\mathcal{Z}$. For the overall expression to have no diffusion, the first term Y_t must offset those surplus shocks (implying a particular loading of dx on $d\mathcal{Z}$) and *must not have any sunspot volatility*. Hence, $\sigma_{x,t} = 0$. This result holds for any degree of price stickiness (any κ) and any inflation volatility (any $\sigma_{\pi,t}$). We revisit the determination of σ_π in Section 4.5.

At first glance, the fact that B_t/P_t evolves locally deterministically seems critical but potentially fragile. In principle, the price level itself could feature a diffusive component, i.e., $dP_t/P_t = \pi_t dt + \sigma_{P,t} dZ_t$ for some $\sigma_{P,t}$ to be determined. However, in typical continuous-time models of price stickiness, such a diffusion does not arise. For example, in our world with Rotemberg price stickiness, we have proved that $\sigma_{P,t} = 0$ (see Appendix B). If firms had such fast-moving prices, they would incur too many price adjustment costs, and this is not optimal. Similarly, in a world with Calvo price stickiness, where price-setting opportunities arrive idiosyncratically at some rate χ, a fraction χdt of firms may change their price over a short time interval dt. This also implies $\sigma_{P,t} = 0$.

In other words, the fact that P_t evolves locally deterministically is a standard outcome of sticky price models. Beyond being an implication of the modeling, $\sigma_{P,t} = 0$ is also deeply reasonable: nominal rigidities should mean that there is some high-enough frequency at which prices don't adjust; in continuous time, that high frequency is the Brownian one. (Furthermore, even if $\sigma_{P,t} \neq 0$ somehow, it could not be some arbitrary equilibrium object that allowed the government debt valuation equation to hold; it would need to be consistent with firms' pricing strategies.)

Debt prices, surplus rules, and discount rate variation. More broadly, one wishes to generalize the insights above to avoid the idea that our result is "knife-edge." In the baseline case, the unit price of debt is fixed at 1 (given it is short-term debt), surplus-to-output ratios are exogenous, and the equilibrium SDF is exactly reciprocal to surpluses (due to log utility). Because of these assumptions, there is no channel that can potentially absorb self-fulfilling demand shocks. The generalizations we pursue in the next subsections relax each of these assumptions one-by-one.

Just to motivate briefly why these extensions matter, let us consider a general model with long-term debt, a potentially endogenous surplus-to-output ratio, and CRRA utility with risk aversion γ. In that case, we show in Appendix A.3 that the present-value formula for aggregate government debt is

$$\frac{Q_t B_t}{P_t} = Y_t \mathbb{E}_t \Big[\int_t^\infty e^{-\rho(u-t)} s_u \Big(\frac{Y_u}{Y_t}\Big)^{1-\gamma} du \Big] \tag{28}$$

Suppose Y_t has some self-fulfilling volatility, via $\sigma_{x,t} \neq 0$. Equation (28) illustrates three possible channels that can absorb this volatility, and thus permit it to exist. First, the long-term debt price Q_t can adjust to shocks; in the baseline model, $Q_t = 1$, and so this was not possible. Second, future surplus-to-output ratios $(s_u)_{u \geq t}$ can be endogenous, through a rule that responds to output and inflation, which allows the present value of surplus-to-output ratios to adjust to shocks. Third, the term $e^{-\rho(u-t)} \big(\frac{Y_u}{Y_t}\big)^{1-\gamma} = \frac{M_u}{M_t} \frac{Y_u}{Y_t}$ represents the net variation of discount rates (i.e., marginal utility growth M_u/M_t) and economic growth (i.e., output growth Y_u/Y_t); in the baseline model, $\gamma = 1$, and this net variation was zero.

Overall, these three extensions are ways in which terms besides Y_t can have diffusive variation. Nevertheless, we will demonstrate in subsequent sections that the key conclusion of Theorem 1 continues to hold, suggesting the logic of why FTPL selects $\sigma_x = 0$ runs deeper than timing assumptions or mathematical artifacts.

Aggregate demand management. So then why, more deeply, do fiscal policies trim equilibria in this manner? One way to obtain an intuition is to consider the rigid price limit $\kappa \to 0$, where government debt becomes equivalent to real debt. Nothing about the analysis above hinges on the value of κ, and so FTPL still selects $\sigma_x = 0$. At first, this may seem confusing because FTPL is thought to be inoperative in a world with inflation-indexed debt, as inflation-indexed debt is thought to render the price level indeterminate (see Chapter 8.1 of Cochrane, 2023). The key is to realize that the government debt valuation equation (GD) becomes a "no-default" condition in a rigid-price world. Rather than determine the price level, or future inflation, it says that surpluses must eventually be positive enough to justify the current debt value. But given the government's exogenous taxation and spending regime, and without the flexibility that inflation provides, the only way a government can fulfill its no-default commitment is if demand takes a particular path. The government debt valuation equation (GD) thus constrains demand.

Our argument is that, surprisingly, a version of this logic extends to any $\kappa > 0$, because FTPL is not really a theory of the price level *per se*, but a theory of aggregate demand management. In fact, "demand management" corresponds to the typical stories told about FTPL. Cochrane (2023), Chapter 2.3, writes

> What force pushes the price level to its equilibrium value? ...If the price level is too low, money may be left overnight. Consumers try to spend this money, raising aggregate demand. Alternatively, a too-low price level may come because the government soaks up too much money from bond sales. Consumers either consume too little today relative to the future or too little overall, violating intertemporal optimization or the transversality condition. Fixing these, consumers again raise aggregate demand, raising the price level.

The key margin of adjustment in these stories is aggregate demand. In a frictionless model, the equilibrium price reflecting this adjustment is the price level. But in sticky price models, the price level cannot jump, so equilibrium partly adjusts via output.

Here, there is a sense in which *entire adjustment to fiscal policy* comes via output. In particular, Theorem 1 says that FTPL eliminates self-fulfilling demand volatility, pins down the response of x to surplus shocks, and provides an "initial condition" that we will ultimately show pins down x_0. Nothing about this story relates to inflation. Instead, as we discuss in Section 4.5, inflation determined by monetary policy.

Generalizing the results: a Markovian class of equilibria. Next, we generalize the key result that $\sigma_x = 0$. We explore (i) long-term debt; (ii) fiscal "rules" rather than exogenous surpluses; and (iii) more general CRRA utility. Because these settings can become substantially more complex, the proofs become unwieldy in the general case.

For that reason, we specialize to the following class of Markovian equilibria and prove our claims within this class.

Definition 3. An *x-Markov equilibrium* is a non-explosive equilibrium such that inflation π_t and the diffusion vector of dx_t are functions of (x_t, s_t, Ω_t), where $s_t := S_t/Y_t$ is the surplus-output ratio and Ω_t are any exogenous states driving s_t.

We conjecture that the equilibria covered by Definition 3 constitute a sufficiently general class for the following reasons. First, it clearly covers the rigid-price limit $\kappa \to 0$, since $\pi_t = 0$ is automatically a function of (x, s, Ω). Because many of our sunspot equilibrium constructions from Section 3 have been derived in this rigid-price limit, we automatically address those. Second, Appendix E presents a volatile sunspot equilibrium with inflation which takes the form $\pi_t = \pi(x_t)$, proving that the x-Markov condition does not by itself rule out sunspot equilibria when $\kappa > 0$. In fact, if FTPL can impose $\sigma_x = 0$ within the class of x-Markov equilibria, then it will have ruled out all the sunspot equilibria constructed in this paper. In this sense, we think the x-Markov class is rich enough to be useful and non-trivial.

4.2 FTPL with long-term debt

One important generalization replaces short-term debt with long-term debt. This is naturally of interest because short-term debt prices can never respond to shocks. This may lead one to think that short-term debt mechanically, in a knife-edge sense, rules out self-fulfilling demand volatility.

To fix ideas and keep things tractable, let us assume that debt is coupon-free and has a constant exponential maturity structure. Per unit of time dt, a constant fraction βdt of outstanding debts mature, and their principal must be repaid. Denote the per-unit price of this debt by Q_t. The government's flow budget constraint is now

$$Q_t \dot{B}_t = \beta B_t - \beta B_t Q_t - P_t S_t. \tag{29}$$

This says that new net debt sales $\dot{B}_t + \beta B_t$, which garner price Q_t, plus primary surpluses $P_t S_t$ must be sufficient to pay back maturing debts βB_t. By standard no-arbitrage asset-pricing, the per-unit bond price is given by

$$Q_t = \mathbb{E}_t \Big[\int_t^\infty \frac{M_T}{M_t} \frac{P_t}{P_T} \beta e^{-\beta(T-t)} dT \Big]. \tag{30}$$

In the above, debt is nominal, so it is priced using the nominal SDF M/P (intuitively, dividing by P converts a nominal cash flow into a real cash flow). The total real value of debt is $Q_t B_t / P_t$, and so the government debt valuation equation is now

$$\frac{Q_t B_t}{P_t} = \mathbb{E}_t \Big[\int_t^\infty \frac{M_u}{M_t} S_u du \Big]. \tag{31}$$

In an equilibrium with long-term debt, all three equations (29), (30), and (31) must hold.

To develop the intuition, we first consider the special example where the interest rate is pegged $\iota_t = \bar{\iota}$. Recall the result from (27) that, if the surplus-to-output ratio s_t follows an arbitrary exogenous process, then the right-hand-side of (31) equals Y_t times an exogenous variable. Equate this expression to $Q_t B_t / P_t$ and apply Itô's formula to both sides, recalling equation (29) for $\dot{B}_t$ and that $\dot{P}_t / P_t = \pi_t$. By matching the "dZ" terms, we obtain

$$\sigma_{Q,t} = \sigma_{x,t}, \tag{32}$$

where σ_Q denotes the loading of $\log(Q_t)$ on dZ_t. In other words, the self-fulfilling demand shocks must be absorbed by long-term debt prices. The key question is whether the pricing of long-term debt in (30) is consistent with this absorption.

Now, to price each bond, note that the nominal SDF in this setting is

$$\frac{M_t}{P_t} = \exp\Big[- \int_0^t \iota_u du - \frac{1}{2} \int_0^t \sigma_{x,u}^2 du - \int_0^t \sigma_{x,u} dZ_u \Big].$$

Using the notation $\tilde{\mathbb{E}}$ for the risk-neutral expectation (which absorbs the martingale $\frac{1}{2} \int_0^t \sigma_{x,u}^2 du - \int_0^t \sigma_{x,u} dZ_u$), the debt price from (30) is then

$$Q_t = \tilde{\mathbb{E}}_t \Big[\int_t^\infty \beta e^{-\int_t^T (\iota_u + \beta) du} dT \Big].$$

Finally, use the assumption of a pegged interest rate $\iota_t = \bar{\iota}$, which implies $Q_t = \frac{\beta}{\bar{\iota} + \beta}$. Debt prices are constant, so $\sigma_Q = 0$, and therefore equation (32) implies $\sigma_x = 0$. In fact, the risk-neutral bond pricing formula just above reveals that the *only way* self-fulfilling demand can enter Q_t is via the interest rate rule. But this suggests that the result is much more general than the peg example: monetary policy would need to follow a very particular rule in order to create fluctuations in the bond price that are consistent with self-fulfilling demand, which generically would not happen.

The conclusions do extend to a more general setting with unpegged interest rates.

In the generalizations we consider, the debt price is no longer constant and can have volatility. However, the volatility implied by the bond pricing equation (30) is inconsistent with the bond price volatility required to support self-fulfilling demand in (31), unless all these volatilities are zero. Essentially, the introduction of long-term debt allows for one extra degree of freedom, namely σ_Q, to absorb self-fulfilling demand shocks, but it also introduces an extra constraint, namely the no-arbitrage pricing equation for a single unit of debt. If σ_Q were some arbitrary process absorbing demand shocks, that would violate the pricing equation for debt.

Theorem 2. *Consider the economy with long-term debt. Suppose equilibrium is x-Markov and either (i) $s_t = \bar{s}$ or (ii) $\kappa \to 0$. Then, the economy necessarily has $\sigma_{x,t} = 0$.*

4.3 FTPL with fiscal rules

Our next generalization allows surpluses to respond to endogenous variables in changes, similarly to the interest rate rule. Suppose again that $S_t = s_t Y_t$, where

$$ds_t = \big(\theta_x x_t + \theta_\pi \pi_t + \theta_s(\bar{s} - s_t)\big)dt + \varsigma_s d\mathcal{Z}_t, \quad \bar{s} > 0, \tag{33}$$

where $\mathcal{Z}$ is a Brownian motion independent of Z. In (33), surpluses respond to both output and inflation, receive shocks, but otherwise mean-revert at rate θ_s back to $\bar{s}$. In this environment, we will also specialize to the linear Taylor rule (linear MP) with target rate $\bar{\iota} = \rho$ to keep the analysis tractable.

Repeating the debt valuation computation from (GD), we obtain

$$\frac{B_t}{P_t} = Y_t \Psi_t, \tag{34}$$

$$\text{where} \quad \Psi_t := \mathbb{E}_t\Big[\int_t^\infty e^{-\rho(T-t)} s_T dT\Big] \tag{35}$$

In the class of x-Markov equilibria of Definition 3, we have the major simplification that $\Psi_t = \Psi(x_t, s_t)$ for some function Ψ that only depends on x_t and s_t. The basic reasoning is as follows. While future surpluses s_T are determined by the entire path of $(x_u, \pi_u, s_u)_{u \in [t,T)}$, an x-Markov equilibrium has the simplifying property that $\sigma_{x,t} = \sigma_x(x_t, s_t)$ and $\pi_t = \pi(x_t, s_t)$ are purely functions of x and s. Together with the surplus rule form in (33), we have that the joint dynamics of (x_t, π_t, s_t) are solely determined by (x_t, s_t). This Markovian property implies that Ψ_t is solely determined by (x_t, s_t).

Even without computing the function Ψ, by applying Itô's formula to (34) and examining the loading on the sunspot shock dZ_t, we can say that

$$0 = \sigma_{x,t}\Big(1 + \frac{\partial}{\partial x}\Psi(x_t, s_t)\Big) \tag{36}$$

One possibility is $\sigma_x = 0$, which the natural case we hope to prove. On the other hand, if $\sigma_{x,t} \neq 0$, then the present-value of future surpluses needs to inherit any output gap volatility, implying a particular functional form for Ψ. What we show is that this functional form is generically inconsistent with equilibrium, which means that $\sigma_{x,t} = 0$ must hold. ("Generically" means the statement holds for almost all parameters.)

Theorem 3. *Consider the economy with fiscal rule (33) and monetary rule (linear MP) with $\bar{\iota} = \rho$. Suppose equilibrium is x-Markov. Suppose either (i) $\kappa \to 0$ or (ii) $\varsigma_s \to 0$ and (x_t, s_t) is a Feller continuous process.*[8] *Then, generically the economy has $\sigma_{x,t} = 0$.*

4.4 FTPL with general CRRA utility

Finally, we replace log utility with general CRRA $u(c, l) = \frac{c^{1-\gamma}}{1-\gamma} - \frac{l^{1+\varphi}}{1+\varphi}$. This extension is of interest because log utility exhibits the knife-edge property that the present-value of future surpluses can have no contribution from "discount rate fluctuations", since the log utility SDF is related to the inverse of output.

In the CRRA world, two changes arise from the new consumption FOC $M_t = e^{-\rho t} Y_t^{-\gamma}$. First, the IS curve now takes the slightly different form (A.9), and it depends on γ. Second, the present value of surpluses is now different: with a constant surplus-to-output ratio $s_t = \bar{s}$, we have

$$\mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t} S_u du\Big] = \bar{s} Y_t \mathbb{E}_t\Big[\int_t^\infty e^{-\rho(u-t)}\Big(\frac{Y_u}{Y_t}\Big)^{1-\gamma} du\Big]$$

The important point relative to log utility is that the present value of surpluses can now admit an additional type of fluctuation, because future discount rates $M_t = e^{-\rho t} Y_t^{-\gamma}$ do not exactly offset surplus growth $S_t = \bar{s} Y_t$. This potentially permits short-run volatility σ_x because it can be absorbed by future discount rates, leaving the present-value of surpluses unaffected. That said, we prove that our key result carries over to CRRA preferences in some cases. The key intuition, similar to the extensions in the main text, is that the absorption of short-run volatility by future discount rates requires a very

[8] Feller continuity means that the transition laws of these processes are continuous in their initial conditions. This condition should be regarded as relatively minor and potentially even superfluous because the equilibrium restrictions in our settings will always imply that x_t and s_t are necessarily path-continuous.

particular specification for the present-value of surpluses, and this specification can be shown to be inconsistent with the other equilibrium conditions.

Theorem 4. *Consider the economy with CRRA utility. Consider equilibria which are x-Markov. Then, the following hold:*

(i) In the rigid-price limit ($\kappa \to 0$), the economy necessarily has $\sigma_{x,t} = 0$.

(ii) If monetary policy follows a linear Taylor rule (linear MP) with target rate $\bar{\iota} = \rho$ and $\rho\phi_x + \kappa(\phi_\pi - 1) < 0$, if the surplus-to-output ratio is constant $s_t = \bar{s}$, if risk aversion is $\gamma > 1$, and if the substitution elasticity ε is not too large, then the long-run equilibrium of the economy necessarily has $\sigma_{x,t} = 0$.

4.5 Determination of inflation

Theorem 1 only provides a "local" result, i.e., that $\sigma_x = 0$, without characterizing the full dynamic equilibrium. It also shows that inflation is not determined from the debt valuation equation alone. Here, we show how monetary policy is needed to pin inflation down. For tractability, we specialize to the following quasi-linear setting. We consider the linearized Phillips curve (linear PC), replacing $(1+\varphi)^{-1}[e^{(1+\varphi)x} - 1] \approx x$ in equation (23). We then specialize the surplus dynamics in (26) to $ds_t = \lambda_s(\bar{s} - s_t) + \varsigma_{s,t} \cdot d\mathcal{Z}_t$, which is a continuous-time version of an AR(1) but with an arbitrary volatility process $\varsigma_{s,t}$. Finally, let us assume the linear Taylor rule (linear MP) with target rate $\bar{\iota}_t = \rho - \frac{1}{2}|\varsigma_{x,t}|^2$, where ς_x is the endogenous sensitivity of x to $d\mathcal{Z}$. (Because Theorem 1 pins down $\varsigma_{x,t}$, its inclusion in the target rate is conceptually distinct from the "risk premium targeting" we studied in Section 3.2. The present target rate only serves as a normalization, so that the economy fluctuates around $(x, \pi) = (0, 0)$.)

Writing the equilibrium dynamics in vector form, with $F_t := (x_t, \pi_t)'$, we have

$$dF_t = \mathcal{A}F_t dt + \mathcal{B}_t dZ_t + \mathcal{C}_t d\mathcal{Z}_t,$$
$$\text{where} \quad \mathcal{A} := \begin{bmatrix} \phi_x & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix}, \quad \mathcal{B}_t := \begin{bmatrix} 0 \\ \sigma_{\pi,t} \end{bmatrix}, \quad \text{and} \quad \mathcal{C}_t := \begin{bmatrix} \varsigma'_{x,t} \\ \varsigma'_{\pi,t} \end{bmatrix}$$

Notice that the first entry of $\mathcal{B}_t$ is zero, because of Theorem 1. Theorem 1 also places some restrictions on the first entry of surplus shock loadings $\mathcal{C}_t$. We follow a relatively standard analysis by doing a spectral decomposition of the transition matrix $\mathcal{A} = V\Lambda V^{-1}$, and analyzing the rotated system $\tilde{F}_t := V^{-1}F_t$. By integrating the system

$d\tilde{F}_t = \Lambda \tilde{F}_t dt + V^{-1}\mathcal{B}_t dZ_t + V^{-1}\mathcal{C}_t \cdot d\mathcal{Z}_t$, we obtain

$$\mathbb{E}_0 \tilde{F}_t = \exp(\Lambda t)\tilde{F}_0. \tag{37}$$

The rest is a familiar stability analysis of (37), in view of the non-explosion Condition 1. There are three cases: both eigenvalues have positive real parts, the eigenvalues have opposite signs, or both eigenvalues have negative real parts. Pursuing this analysis, we then obtain the following generalization of some familiar results, with the proof in Appendix A.5.

Proposition 6. *Consider the linearized Phillips curve (linear PC), the linear Taylor rule (linear MP) with target rate $\bar{\iota} = \rho + \frac{1}{2}|\varsigma_{x,t}|^2$, and the surplus dynamics $ds_t = \lambda_s(\bar{s} - s_t) + \varsigma_{s,t} \cdot d\mathcal{Z}_t$. A non-explosive equilibrium takes one of three forms, ignoring knife-edge cases for the parameters:*

1. *If $\rho + \phi_x > 0$ and $\rho\phi_x + \kappa(\phi_\pi - 1) > 0$, then equilibrium generically fails to exist.*
2. *If $\rho\phi_x + \kappa(\phi_\pi - 1) < 0$, the unique equilibrium features $\pi_t = \frac{1}{2}\frac{\rho - \phi_x - \sqrt{(\rho-\phi_x)^2 - 4\kappa(\phi_\pi - 1)}}{\phi_\pi - 1} x_t$ with $\sigma_{x,t} = \sigma_{\pi,t} = 0$.*
3. *If $\rho + \phi_x < 0$ and $\rho\phi_x + \kappa(\phi_\pi - 1) > 0$, then the equilibrium is not unique.*

Proposition 6 is reminiscent of the large literature of FTPL in linearized New Keynesian models. Equilibrium cannot exist with both "active fiscal" and "active money" regimes (case 1). Equilibrium exists and is unique when "passive money" is paired with "active fiscal" (case 2). These results echo Leeper (1991). A finding which differs slightly from the literature is our case 3: monetary policy that acts super aggressively against inflation but acts counterintuitively to output induces non-uniqueness, despite active fiscal policy. This case can have $\sigma_\pi \neq 0$ because monetary policy induces globally stable dynamics through its rule. This third case shows clearly that monetary policy remains, in fact, critical to inflation determination, even when FTPL is operative. Of these three, the interesting case is the active-fiscal passive-money regime (case 2), which delivers a unique equilibrium.

The most important takeaway from Proposition 6 is that FTPL eliminates self-fulfilling fluctuations for a broad range of monetary policy rules. FTPL does not need a single knife-edge monetary rule to do this. As long as $\rho\phi_x + \kappa(\phi_\pi - 1) < 0$ (monetary policy is not too aggressive), FTPL guarantees $\sigma_x = \sigma_\pi = 0$. For instance, an interest rate peg fits into this case, and delivers a unique equilibrium converging to steady state (absent surplus shocks) at rate $\frac{1}{2}[\sqrt{\rho^2 + 4\kappa} - \rho]$. The absence of self-fulfilling fluctuations contrasts sharply with the results of Section 3, where without active fiscal policy, sunspot volatility could exist *for any conventional monetary policy rule*.

Theorem 1 with Proposition 6 provide a clean breakdown of what fiscal and monetary policies do. Fiscal policy provides aggregate demand management that rules out real sunspot volatility (i.e., $\sigma_x = 0$) and induces surplus shocks to become demand shocks (i.e., ς_x is pinned down), for any monetary policy rule. The monetary policy rule (i.e., ϕ_x, ϕ_π) then connects inflation to output. For example, in the standard case 2, monetary policy forces π_t to be a function of x_t, so that sunspot inflation volatility is zero (i.e., $\sigma_\pi = 0$) and inflation shocks are fiscally determined (i.e., ς_π is pinned down by ς_x).[9]

5 Conclusion

We show that New Keynesian models inherently permit a novel type of sunspot volatility that appears only in the nonlinear version of the model. The distinguishing features of our volatility are that it is self-fulfilled by the presence of risk premia and can arise only in recessionary times. While conventional monetary policy has almost no power to trim these volatile equilibria, active fiscal policies do so across a wide variety of settings. Our fiscal theory examples permit: any level of price stickiness, long-term debt, arbitrary exogenous surpluses, and some types of endogenous surpluses (surplus rules).

What are the implications of our results for current practices in monetary economics? Importantly, the standard New Keynesian paradigm of using the Taylor principle to select a unique equilibrium is not necessarily valid when considering risk and risk premia. By contrast, the FTPL approach has merit, implying monetary and fiscal policies are not *alternatives* to each other in selecting a unique equilibrium. Rather, active fiscal policy provides determinacy in strictly more situations than active money. But here, FTPL operates differently than conventionally thought. In our nonlinear, stochastic solution, FTPL works by eliminating the self-fulfilling volatility in real variables (i.e., $\sigma_x = 0$), while the monetary policy rule is needed to determine inflation. Thus, our analysis provides a clean distinction between the role of fiscal and monetary policies, a distinction that is not evident in existing research that restricts attention to linear equilibria.

Finally, while we have considered a stripped-down version of the New Keynesian model, we expect FTPL to work similarly with more features. Taking our results as suggestive, how should researchers apply FTPL to more complex environments? As

[9] The reader may notice that we have paid little attention to the "initial condition" provided by FTPL. In particular, Theorem 1 says that FTPL selects $\sigma_{x,t} = 0$ for all $t \geq 0$, requires that (GD) hold at $t = 0$, and does nothing more. The requirement that (GD) hold at $t = 0$ disciplines the initial demand level x_0, rather than disciplining anything about future dynamics (x_t, π_t). Indeed, consider the standard case 2 of Proposition 6, which says that the monetary policy rule exclusively determines π_t, given x_t. In that case, the IS curve, Phillips curve, Taylor rule, and surplus dynamics fully determine the dynamics of (x_t, π_t), conditional on x_0. The initial condition of (GD) at $t = 0$ then pins down x_0.

a first step, one ignores any self-fulfilling real shocks (e.g., $\sigma_x = 0$). Then, one pairs fiscal policy with "passive" monetary policy and solves for a Markov equilibrium where inflation π and other forward looking variables are functions of the output gap x and any other state variables Ω. This is the natural outcome in our more special setting, which we conjecture would carry over to more complex models.

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Online Appendix:
Fear, Indeterminacy, and Policy Responses

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March 14, 2025

A Proofs

A.1 Non-explosion and transversality conditions

For completeness, we briefly document the non-explosion requirements imposed by consumer and firm optimality. We then show that our non-explosion Condition 1 suffices to ensure these requirements hold. Thus, besides the standard derivations in the text, this completes the proof of Lemma 1.

For the consumer side, note that the representative agent's utility can be written

$$U_0 = \rho^{-1}\Big(\log Y^* - \frac{(Y^*)^{1+\varphi}}{1+\varphi}\Big) + \int_0^\infty e^{-\rho t}\mathbb{E}\Big[x_t - \frac{e^{(1+\varphi)x_t}}{1+\varphi}\Big]dt$$

We need to ensure the consumer obtains finite utility and that his transversality condition holds. To ensure $U_0 > -\infty$, we require

$$\lim_{T\to\infty} \mathbb{E}_t[e^{-\rho T}x_T] = 0 \tag{A.1}$$

$$\lim_{T\to\infty} \mathbb{E}_t[e^{(1+\varphi)x_T-\rho T}] = 0 \tag{A.2}$$

Requirement (A.1) rules out $\mathbb{E}x_T$ diverging to $-\infty$ faster than rate ρ. Requirement (A.2) rules out $\mathbb{E}e^{(1+\varphi)x_T}$ diverging to $+\infty$ faster than rate ρ. It is clear that if Condition 1 holds, then both (A.1)-(A.2) are satisfied.

The consumer's transversality condition holds if and only if the lifetime budget constraint (5) holds with equality. Now, note that since price adjustment costs are non-pecuniary, the real present value of aggregate profits are $\Pi_t = \mathbb{E}_t[\int_t^\infty \frac{M_s}{M_t}(Y_s - \frac{W_sL_s}{P_s})ds]$. Using the resource constraint $C_t = Y_t$ and $B_0 = 0$, we therefore have that the consumer lifetime budget constraint (5) holds with equality, so long as all these integrals converge. Convergence of the integrals can be evaluated using the FOCs. The consumption FOC (8) implies $\mathbb{E}_0[\int_0^\infty M_tC_tdt] = (\rho\lambda)^{-1}$, so this integral converges. The labor FOC (7) and market clearing $C_t = L_t$ imply $\mathbb{E}_0[\int_0^\infty M_t\frac{W_tL_t}{P_t}dt] = \lambda^{-1}\mathbb{E}_0[\int_0^\infty e^{-\rho t}C_t^{1+\varphi}dt]$, so this integral converges so long as $\mathbb{E}[C_t^{1+\varphi}]$ grows slower than $e^{\rho t}$, which is exactly identical to requirement (A.2) that has already been verified.

For the firm side, note that Appendix B derives the optimality conditions from the firm's price setting problem. There, we show that the firm's transversality conditions are, in a symmetric equilibrium in which firms charge identical prices,

$$\lim_{T\to\infty} \mathbb{E}_t[e^{(1+\varphi)x_T-\rho T}] = 0 \tag{A.3}$$

$$\lim_{T\to\infty} \mathbb{E}_t[e^{-\rho T}\pi_T^2] = 0 \tag{A.4}$$

Notice that requirement (A.3) is identical to (A.2), which we have already verified. Requirement (A.4) avoids nominal explosions that imply an infinite present value of adjustment costs. Note that under Condition 1, this automatically holds.

A.2 Proofs for Sections 2-3

Proof of Proposition 1. See the proof of Proposition 6 in Appendix A.5 for the spectral decomposition of the matrix $\mathcal{A}$. Asymptotic instability of this system is guaranteed if $\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2) > 0$. This holds if and only if $\det(\mathcal{A}) > 0$ and $\mathrm{tr}(\mathcal{A}) > 0$, which is equivalent to $\phi_x > -\rho$ and $\phi_\pi > 1 - \rho\phi_x/\kappa$. □

Proof of Proposition 2. Since inflation is rigid, consider any Taylor rule with target rate $\bar{\iota} = \rho$ and response function $\Phi(x)$ that is increasing in x. The dynamics of $y_t = e^{x_t}$ are given by

$$dy_t = y_t\Big[\Phi(\log(y_t)) + \sigma_{x,t}^2\Big]dt + y_t\sigma_{x,t}dZ_t.$$

Let $\Phi(x)$ satisfy $\lim_{x\to-\infty} e^{\hat{\beta}x}\Phi(x) > -\infty$ for some $\hat{\beta} > 0$, as required by Condition 2. Define $\beta := \max(1, \hat{\beta})$. Obviously, we also have $\lim_{x\to-\infty} e^{\beta x}\Phi(x) > -\infty$.

Specify volatility by, for any $\nu > 0$ and $y_{\max} < 1$,

$$\sigma_x^2 = \begin{cases} \frac{2}{1+\beta}\left[y^{-2\beta}\nu^2 - \Phi(\log(y))\right], & \text{if } y < y_{\max}; \\ 0, & \text{if } y \geq y_{\max}. \end{cases} \tag{A.5}$$

Since $\Phi(\cdot)$ is increasing and continuous, and $\Phi(0) = 0$, we have that $y^{-2\beta}\nu^2 > \Phi(\log(y))$ for all $y < y_{\max}$, as needed to ensure $\sigma_x^2 \geq 0$. From here, a similar argument applies as in the text but for the process y_t^β rather than y_t.

By Itô's formula, and then using (A.5),

$$
\begin{aligned}
dy_t^\beta &= \beta y_t^\beta \Big(\Phi(\log(y_t)) + \frac{\beta+1}{2}\sigma_{x,t}^2 \Big) dt + \beta y_t^\beta \sigma_{x,t} dZ_t \\
&= \beta \Big[y_t^\beta \Phi(\log(y_t)) \mathbf{1}_{\{y_t \geq y_{\max}\}} + \frac{\nu^2}{y_t^\beta} \mathbf{1}_{\{y_t < y_{\max}\}} \Big] dt + \beta y_t^\beta \sigma_{x,t} dZ_t
\end{aligned}
$$

Note that

$$
\lim_{y \to 0} y^{2\beta} \sigma_x^2(y) = \frac{2}{1+\beta} \Big(\nu^2 - \lim_{y \to 0} y^{2\beta} \Phi(\log(y)) \Big) = \frac{2\nu^2}{1+\beta}
$$

the latter equality because $y^\beta \Phi(\log(y))$ finite as $y \to 0$, implying $y^{2\beta}\Phi(\log(y)) \to 0$. Therefore, the dynamics of y_t^β coincide with a Bessel(n) process asymptotically as $y_t \to 0$, where $n := 2 + \beta^{-1}$.[10] Because $\beta \in [1, \infty)$, we have $n \in (2, 3]$. By properties of the Bessel(n) process, the boundary $\{y = 0\}$ is an entrance boundary, hence inaccessible, for any $n > 2$.

On the other hand, for $y \in [y_{\max}, 1)$, we have $dy_t^\beta = \beta y_t^\beta \Phi(\log(y_t)) dt < 0$, by the fact that $\Phi(0) = 0$ and $\Phi(x)$ is increasing. Therefore, y_t enters the region $(-\infty, y_{\max})$ in finite time when starting from any point $y_0 \leq 1$. Putting together the boundary behavior at these two endpoints, and noting that y_t^β has non-zero diffusion in the interior, we have that y_t^β has a non-degenerate stationary distribution. Hence, y_t also possesses a non-degenerate stationary distribution, and its ergodic set is $(-\infty, y_{\max}]$.

Finally, we prove the claim that any volatility function is valid if it satisfies suitable boundary conditions. Instead of the σ_x function in (A.5), consider any alternative function $\tilde{\sigma}_x$, which (i) coincides with σ_x for $x \notin (-C, -C^{-1})$ for C arbitrarily large; and (ii) is bounded on $x \in (-C, -C^{-1})$. By inspection, the entire proof above remains valid. □

Proof of Proposition 3. Again, inflation is rigid. Consider the standard linear rule with $\bar{\imath} = \rho$ and $\Phi(x) = \phi_x x$, with $\phi_x > 0$. This rule satisfies Condition 2, and hence

[10] Indeed, we have

$$
dy^\beta \sim \frac{\beta \nu^2}{y^\beta} dt + \beta \nu \sqrt{\frac{2}{1+\beta}} dZ \quad \text{as} \quad y \sim 0.
$$

Define $\tilde{y} := y^\beta (\frac{1+\beta}{2})^{1/2} \frac{1}{\beta \nu}$. Then,

$$
d\tilde{y} \sim (\frac{1+\beta}{2\beta}) \frac{1}{\tilde{y}} dt + dZ, \quad \text{as} \quad y \sim 0.
$$

A Bessel(n) process takes the form $d\tilde{y} = \frac{n-1}{2} \tilde{y}^{-1} dt + dZ$, which the above process matches by picking $n = 2 + \beta^{-1}$.

volatile equilibria exist. It remains to prove that $x_t \leq 0$ in any equilibrium.

To do this, suppose $x_0 > 0$, leading to contradiction. Suppose $\sigma_{x,t}$ is any arbitrary equilibrium volatility process. Then, the output gap follows

$$dx_t = \Big(\phi_x x_t + \frac{1}{2}\sigma_{x,t}^2\Big)dt + \sigma_{x,t} dZ_t$$

Consider the auxiliary process

$$d\tilde{x}_t = \phi_x \tilde{x}_t dt + \sigma_{x,t} dZ_t, \quad \tilde{x}_0 = x_0,$$

which features an identical initial condition and shock exposure as x but without the risk premium in the drift. By standard comparison theorems for diffusions, we thus have $\mathbb{E}_0[\tilde{x}_T] \leq \mathbb{E}_0[x_T]$. We may solve the SDE for $\tilde{x}$ as

$$\tilde{x}_T = e^{\phi_x T}\tilde{x}_0 + \int_0^T e^{\phi_x(T-t)}\sigma_{x,t} dZ_t$$
$$\Rightarrow \lim_{T\to\infty} \mathbb{E}_0[\tilde{x}_T] = \lim_{T\to\infty} e^{\phi_x T}\tilde{x}_0 = +\infty.$$

Thus, $\mathbb{E}_0[x_T] \to +\infty$ in violation of non-explosion Condition 1. $\square$

Proof of Proposition 4. It suffices to prove the proposition in the case $\alpha_- = \alpha_+ = 1$, because when $\alpha_- > 1 > \alpha_+$, the drift of dx_t is increased (decreased) when x_t is positive (negative). And hence the dynamics push x_t further away from zero than they would in the case $\alpha_- = \alpha_+ = 1$. (Formally, standard diffusion comparison theorems imply that $|x_t|$ will be forever further from zero, almost surely, than it would in the case $\alpha_- = \alpha_+ = 1$.)

If $\alpha_- = \alpha_+ = 1$, then the dynamics of $y_t = e^{x_t}$ in the rigid-price limit are given by

$$dy_t = y_t \Phi(\log(y_t))dt + y_t \sigma_x(\log(y_t))dZ_t$$

Recall the assumption that the volatility $y_t\sigma_x(\log(y_t))$ remains bounded as $y_t \to 0$. Write $\bar{\sigma} := \lim_{y\to 0} y\sigma_x(\log(y))$. Choose $\Phi(x) = \frac{\phi_x}{2}(e^x - e^{-x})$. Then, asymptotically as $y \to 0$, the drift of y_t is equal to $-\frac{\phi_x}{2}$ and the volatility equal to $\bar{\sigma}$. This asymptotic behavior is exactly identical to an arithmetic Brownian motion. Hence, as long as $\phi_x > 0$, then $y_t \to 0$ in finite time with positive probability, violating Condition 1. And so we must have $\sigma_x = 0$ when $y < 1$. By examining the dynamics of $\tilde{y}_t := 1/y_t$,

$$d\tilde{y}_t = -\tilde{y}_t[\Phi(-\log(\tilde{y}_t)) + \sigma_x(-\log(\tilde{y}_t))^2]dt - \tilde{y}_t\sigma_x(-\log(\tilde{y}_t))dZ_t,$$

and using the assumption that volatility $\tilde{y}_t\sigma_x(-\log(\tilde{y}_t))$ remains bounded as $\tilde{y}_t \to 0$, we can argue analogously that $\tilde{y}_t \to 0$ in finite time with positive probability. Together, these arguments imply the unique non-explosive equilibrium is $y_t = 1$, hence $x_t = 0$ forever. □

A.3 Characterization of FTPL in a general setting

Before proving the theorems from the text, we derive a useful characterization of the government debt valuation equation that holds in a general environment nesting all the cases in the text. The environment below will feature a general surplus process, long-term debt, and CRRA utility.

We first set up a general surplus dynamic that nests all cases of interest. Let $\mathcal{Z}_t$ be a k-dimensional Brownian motion independent of the sunspot shock Z_t. Let Ω follow a Markov diffusion driven by $\mathcal{Z}$. Let the surplus-to-output ratio $s_t := S_t/Y_t$ follow a process of the form

$$ds_t = \mu_s(\Omega_t, x_t, \pi_t)dt + \varsigma_s(\Omega_t, x_t, \pi_t) \cdot d\mathcal{Z}_t \tag{A.6}$$

$$d\Omega_t = \mu_\Omega(\Omega_t)dt + \varsigma_\Omega(\Omega_t) \cdot d\mathcal{Z}_t \tag{A.7}$$

For now, we let the drifts and diffusions μ_s, μ_Ω, ς_s, ς_Ω be arbitrary functions. This is more general than what we need going forward. Note that we obtain exogenous surpluses, following the description in the text before equation (26), if we impose that μ_s and ς_s only depend on Ω. Furthermore, we obtain surplus rules if we pick the dependence of μ_s on (x, π) appropriately.

Second, we generalize the model to the CRRA utility $u(c, l) = \frac{c^{1-\gamma}}{1-\gamma} + \frac{l^{1+\varphi}}{1+\varphi}$ as in Section 4.4. In that case, the consumption FOC says

$$M_t = e^{-\rho t}C_t^{-\gamma}. \tag{A.8}$$

The labor-consumption margin is unaffected. Applying Itô's formula to (A.8), and noting that $C_t = Y_t = Y^* e^{x_t}$ and $-\frac{1}{dt}\mathbb{E}[\frac{dM_t}{M_t}] = r_t = \iota_t - \pi_t$, the IS curve generalizes to

$$dx_t = \Big[\frac{\iota_t - \pi_t - \rho}{\gamma} + \frac{1}{2}\gamma\sigma_{x,t}^2 + \frac{1}{2}\gamma|\varsigma_{x,t}|^2\Big]dt + \sigma_{x,t}dZ_t + \varsigma_{x,t} \cdot d\mathcal{Z}_t. \tag{A.9}$$

The dynamics of $Y_t = Y^* e^{x_t}$ can be derived from (A.9). When $\gamma \neq 1$, the Phillips curve is also different and requires an additional approximation to obtain a form similar to that used throughout the paper. Indeed, the derivation of the Phillips curve in Appendix

B relies on $M_t Y_t \propto e^{-\rho t}$, which is no longer true with general CRRA utility. We make this approximation, which is tantamount to approximating around steady-state where $Y_t / Y_0 \approx 1$. With this approximation, the Phillips curve (PC) is replaced by

$$\mu_{\pi,t} = \rho \pi_t - \kappa \Big(\frac{e^{(\gamma+\varphi)x_t} - 1}{\gamma + \varphi} \Big), \tag{A.10}$$

where $Y^* := (\frac{\varepsilon-1}{\varepsilon})^{\frac{1}{\gamma+\varphi}}$ is the flexible-price output level and $\kappa := \eta(\varepsilon-1)(\gamma+\varphi)$ is the composite price-stickiness parameter.

Third, we generalize to the long-term debt setup described in Section 4.2. Let Q_t denote the per-unit bond price, which has dynamics of the form

$$dQ_t = Q_t \Big[\mu_{Q,t} dt + \sigma_{Q,t} dZ_t + \varsigma_{Q,t} \cdot d\mathcal{Z}_t \Big] \tag{A.11}$$

for some μ_Q, σ_Q, and ς_Q to be determined. With long-term debt, the flow government budget constraint is (29), the per-unit bond pricing equation is (30), and the government debt valuation equation is (31). Substituting the consumption FOC (A.8) into (31), we may rewrite the aggregate debt valuation equation as

$$\frac{Q_t B_t}{P_t} = Y_t^{\gamma} \Psi_t \quad \text{where} \quad \Psi_t := \mathbb{E}_t \Big[\int_t^{\infty} e^{-\rho(u-t)} s_u Y_u^{1-\gamma} du \Big] \tag{A.12}$$

The next steps are to derive the dynamics of the two key present values Q_t and Ψ_t.

Starting from the per-unit bond pricing equation (30), we have that the object

$$e^{-\beta t} \frac{Q_t M_t}{P_t} + \int_0^t \frac{M_u}{P_u} \beta e^{-\beta u} du$$

is a local martingale and has zero drift. Note that, from the consumption FOC (A.8), the nominal SDF M_t / P_t has dynamics

$$d(M_t/P_t) = -(M_t/P_t) \Big[\iota_t dt + \gamma \sigma_{x,t} dZ_t + \gamma \varsigma_{x,t} \cdot d\mathcal{Z}_t \Big] \tag{A.13}$$

Then, by applying Itô's formula to the previous expression, and setting the resulting drift to zero, we have

$$\mu_{Q,t} = \beta - \frac{\beta}{Q_t} + \iota_t + \gamma \sigma_{x,t} \sigma_{Q,t} + \gamma \varsigma_{x,t} \cdot \varsigma_{Q,t} \tag{A.14}$$

From the definition of Ψ_t, we have

$$e^{-\rho t}\Psi_t + \int_0^t e^{-\rho u} s_u Y_u^{1-\gamma} du = \mathbb{E}_t\Big[\int_0^\infty e^{-\rho u} s_u Y_u^{1-\gamma} du\Big],$$

which is a local martingale. By the martingale representation theorem, we have that

$$d\Big(e^{-\rho t}\Psi_t + \int_0^t e^{-\rho u} s_u Y_u^{1-\gamma} du\Big) = e^{-\rho t}\Big(\sigma_{\Psi,t} dZ_t + \varsigma_{\Psi,t}\cdot d\mathcal{Z}_t\Big)$$

for some $\sigma_{\Psi,t}$ and some $\varsigma_{\Psi,t}$. On the other hand, we also have by applying Itô's formula to the left-hand-side,

$$d\Big(e^{-\rho t}\Psi_t + \int_0^t e^{-\rho u} s_u Y_u^{1-\gamma} du\Big) = \Big[-\rho e^{-\rho t}\Psi_t + e^{-\rho t} s_t Y_t^{1-\gamma}\Big]dt + e^{-\rho t} d\Psi_t$$

Equating these last two results, and rearranging for $d\Psi_t$, we have

$$d\Psi_t = (\rho\Psi_t - s_t Y_t^{1-\gamma})dt + \sigma_{\Psi,t} dZ_t + \varsigma_{\Psi,t}\cdot d\mathcal{Z}_t \tag{A.15}$$

We now state and prove a useful characterization lemma.

Lemma A.1. *In the setting above with general surpluses, long-term debt, and CRRA utility,*

$$\Psi_t\gamma\sigma_{x,t} = \Psi_t\sigma_{Q,t} - \sigma_{\Psi,t} \tag{A.16}$$

$$\Psi_t\gamma\varsigma_{x,t} = \Psi_t\varsigma_{Q,t} - \varsigma_{\Psi,t} \tag{A.17}$$

Conversely, if the asset-pricing equations (A.14)-(A.15) hold, and the diffusion-matching equations (A.16)-(A.17) hold, then the government debt valuation equation (A.12) holds at every date, provided it holds at the initial date.

Proof of Lemma A.1. We apply Itô's formula to both sides of (A.12), using the flow government budget constraint (29), the price level dynamics $dP_t/P_t = \pi_t dt$, the dynamics of x_t in (A.9), the dynamics of Ψ_t in (A.15), and the dynamics of Q_t in (A.11) and (A.14).

Matching drift and diffusion coefficients, we obtain

$$
\begin{aligned}
[dt]\;:\;\frac{Q_tB_t}{P_t}\big[\iota_t-\pi_t+\gamma\sigma_{x,t}\sigma_{Q,t}+\gamma\varsigma_{x,t}\cdot\varsigma_{Q,t}\big]-s_tY_t
&=Y_t^{\gamma}(\rho\Psi_t-s_tY_t^{1-\gamma})+\gamma Y_t^{\gamma}\Psi_t\Big[\frac{\iota_t-\pi_t-\rho}{\gamma}+\frac{1}{2}(\gamma+1)(\sigma_{x,t}^2+|\varsigma_{x,t}|^2)\Big]\\
&\quad+\frac{1}{2}\gamma(\gamma-1)Y_t^{\gamma}\Psi_t(\sigma_{x,t}^2+|\varsigma_{x,t}|^2)+\gamma Y_t^{\gamma}\sigma_{x,t}\sigma_{\Psi,t}+\gamma Y_t^{\gamma}\varsigma_{x,t}\cdot\varsigma_{\Psi,t}\\
[dZ]\;:\;\frac{Q_tB_t}{P_t}\sigma_{Q,t}&=\gamma Y_t^{\gamma}\Psi_t\sigma_{x,t}+Y_t^{\gamma}\sigma_{\Psi,t}\\
[d\mathcal{Z}]\;:\;\frac{Q_tB_t}{P_t}\varsigma_{Q,t}&=\gamma Y_t^{\gamma}\Psi_t\varsigma_{x,t}+Y_t^{\gamma}\varsigma_{\Psi,t}
\end{aligned}
$$

Equations $[dZ]$ and $[d\mathcal{Z}]$, combined with (A.12), imply (A.16)-(A.17).

Conversely, plugging (A.16)-(A.17) into the first equation $[dt]$, using (A.12), and simplifying, we obtain an identity. Therefore, the $[dt]$ equation holds automatically, given the other equations all hold. This means that, provided (A.12) holds at $t=0$, it will hold at every future date $t>0$. □

A.4 Proofs of FTPL Theorems 1-4

Proof of Theorem 1. We specialize the result of Lemma A.1 as follows. First, with log utility ($\gamma=1$), the present value Ψ_t in (A.12) becomes

$$\Psi_t := \mathbb{E}_t\Big[\int_t^{\infty}e^{-\rho(u-t)}s_u du\Big]$$

Second, with the exogenous Markovian surplus process $s_t=s(\Omega_t)$, we have that Ψ_t is purely determined by Ω_t, i.e., there exists a deterministic function $\Psi(\cdot)$ such that $\Psi_t=\Psi(\Omega_t)$. In that case, we have by Itô's formula and (A.7) that $\sigma_{\Psi,t}=0$. Third, we have instantaneously-maturing debt, which is nested in the above formulas by setting $Q_t=1$. This implies $\sigma_{Q,t}=0$. Using these results, (A.16) holds if and only if $\Psi_t\sigma_{x,t}=0$. Thus, $\sigma_{x,t}=0$ for almost all t (except at the times when $\Psi_t=0$, which are zero Lebesgue measure almost-surely). For the statement about (GD) holding for every $t>0$, given it holds at $t=0$, see the final statement of Lemma A.1. □

Remark A.1 (Non-Markovian surpluses)**.** *From the proof of Theorem 1, it is clear that the same arguments hold even in the more general non-Markovian case where $(s_t)_{t\geq 0}$ is independent of $(Z_t)_{t\geq 0}$, because in that case $\sigma_{\Psi,t}=0$ still holds.*

Proof of Theorem 2. The strategy of the proof is as follows. First, we characterize the general way in which the bond price Q absorbs demand shocks to x, i.e., how the bond price function must look for such absorption to occur. Then, we show that the dynamic pricing equation for Q can only be consistent with the required function if inflation π takes a particular functional form. Finally, we show that the required inflation function cannot be consistent with the Phillips curve unless the monetary policy rule is a knife-edge function. We need to prove the result for the constant surplus-output limit ($s_t \to \bar{s}$) and the rigid-price limit ($\kappa \to 0$). We proceed in a general way that nests both cases and then specialize at the end of the proof.

We start by specializing the result of Lemma A.1 as follows. As in Theorem 1, the combination of log utility ($\gamma = 1$) and exogenous Markovian surplus process $s_t = s(\Omega_t)$, implies that $\Psi_t = \Psi(\Omega_t)$ for some function $\Psi(\cdot)$, and in particular that $\sigma_{\Psi,t} = 0$.

Next, we use the x-Markov assumption (Definition 3) given in the theorem. Recall the bond pricing equation (30), which after plugging in the nominal SDF from (A.13) says

$$Q_t = \mathbb{E}_t\Big[\int_t^\infty e^{-\int_t^u (\iota_\tau + \frac{1}{2}\sigma_{x,\tau}^2)d\tau - \int_t^u \sigma_{x,\tau} dZ_\tau} \beta e^{-\beta(u-t)} du\Big].$$

In an x-Markov equilibrium, we have that $\sigma_{x,t} = \sigma_x(x_t, \Omega_t)$, $\varsigma_{x,t} = \varsigma_x(x_t, \Omega_t)$, and $\pi_t = \pi(x_t, \Omega_t)$ for some functions σ_x, ς_x, and π. In that case, the nominal interest rate $\iota_t = \bar{\iota} + \Phi(x_t, \pi_t) = \bar{\iota} + \Phi(x_t, \pi(x_t, \Omega_t))$ is purely a function of (x_t, Ω_t). These facts imply furthermore that (x_t, Ω_t) is a Markov diffusion (i.e., their dynamics only depend on x_t and Ω_t). Consequently, the bond pricing equation above implies that Q_t is purely a function of (x_t, Ω_t), i.e., $Q_t = Q(x_t, \Omega_t)$ for some function Q to be determined.

Apply Itô's formula to Q to obtain (after dropping t subscripts)

$$\sigma_Q = \sigma_x \partial_x \log Q \tag{A.18}$$

$$\varsigma_Q = \varsigma_x \partial_x \log Q + \varsigma_\Omega \partial_\Omega \log Q \tag{A.19}$$

$$\mu_Q = \frac{1}{Q}\Big[\mu_x \partial_x Q + \mu_\Omega' \partial_\Omega Q + \frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2)\partial_{xx} Q + \frac{1}{2}\mathrm{tr}[\varsigma_\Omega' \varsigma_\Omega \partial_{\Omega\Omega} Q] + \varsigma_x' \varsigma_\Omega \partial_{\Omega x} Q\Big] \tag{A.20}$$

Combining these results with equations (A.14) and (A.16)-(A.17) for μ_Q, σ_Q, and ς_Q (and using the facts that $\sigma_\Psi = 0$, that $\Psi \neq 0$ almost-surely, and that $\gamma = 1$), we obtain

$$\sigma_x = \sigma_x \partial_x \log Q \tag{A.21}$$

$$\varsigma_x + \frac{\varsigma_\Psi}{\Psi} = \varsigma_x \partial_x \log Q + \varsigma_\Omega \partial_\Omega \log Q \tag{A.22}$$

$$\beta - \frac{\beta}{Q} + \iota + \sigma_x^2 + |\varsigma_x|^2 + \frac{\varsigma_x \cdot \varsigma_\Psi}{\Psi} = \frac{1}{Q}\Big(\mu_x \partial_x Q + \mu_\Omega \cdot \partial_\Omega Q\Big) \tag{A.23}$$
$$\frac{1}{Q}\Big[\frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2)\partial_{xx} Q + \frac{1}{2}\mathrm{tr}[\varsigma_\Omega' \varsigma_\Omega \partial_{\Omega\Omega} Q] + \varsigma_x' \varsigma_\Omega \partial_{\Omega x} Q\Big]$$

We will use these equations to determine σ_x.

Let us assume, leading to contradiction, that $\sigma_x \neq 0$. For (A.21) to hold given the assumption that $\sigma_x \neq 0$, it must be that $\partial_x \log Q = 1$, and so

$$Q(x, \Omega) = \bar{Q}(\Omega) e^x$$

for some function $\bar{Q}$. (Technically, this equation holds at all values of x, Ω such that $\sigma_x \neq 0$.) From (A.22), we then have that $\varsigma_\Psi / \Psi = \varsigma_\Omega \partial_\Omega \log \bar{Q}$. Note that this pins down, up to a constant, the function for $\bar{Q}$, given that Ψ and ς_Ω are exogenous. The constant is pinned down by the initial condition for the aggregate debt valuation equation (A.12). Using these results in (A.23), we have

$$\beta - \frac{\beta}{Q} + \iota + \sigma_x^2 + |\varsigma_x|^2 + \varsigma_x' \varsigma_\Omega \partial_\Omega \log \bar{Q} = \mu_x + \frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2)$$
$$+ \mu_\Omega \cdot \partial_\Omega \log \bar{Q} + \frac{1}{2}\frac{\mathrm{tr}[\varsigma_\Omega' \varsigma_\Omega \partial_{\Omega\Omega} \bar{Q}]}{\bar{Q}} + \varsigma_x' \varsigma_\Omega \partial_\Omega \log \bar{Q}$$

Substituting μ_x from (A.9) with $\gamma = 1$, and simplifying, we get

$$(\beta + \rho + \pi)\bar{Q} - e^{-x}\beta = \mu_\Omega \cdot \partial_\Omega \bar{Q} + \frac{1}{2}\mathrm{tr}[\varsigma_\Omega' \varsigma_\Omega \partial_{\Omega\Omega} \bar{Q}]$$

For this last equation to hold, it must be the case that

$$\pi = \frac{\beta}{\bar{Q}} e^{-x} + \bar{\pi} - \beta - \rho \tag{A.24}$$

$$\text{where} \quad \bar{\pi} := \frac{1}{\bar{Q}}\Big(\mu_\Omega \cdot \partial_\Omega \bar{Q} + \frac{1}{2}\mathrm{tr}[\varsigma_\Omega' \varsigma_\Omega \partial_{\Omega\Omega} \bar{Q}]\Big) \tag{A.25}$$

i.e., inflation must be a particular function of (x, Ω). Note that $\bar{\pi}$ is independent of x. This proves the result for the rigid-price limit ($\kappa \to 0$), since $\pi = 0$ must prevail there, in contradiction to equation (A.24). Hence, $\sigma_x = 0$ must hold for this case.

On the other hand, apply Itô's formula to $\pi(x, \Omega)$ and equate the drift to (A.10), denoting $f(x) := \frac{e^{(1+\varphi)x}-1}{1+\varphi}$,

$$\rho\pi - \kappa f(x) = \mu_x \partial_x \pi + \mu_\Omega \cdot \partial_\Omega \pi + \frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2)\partial_{xx}\pi + \frac{1}{2}\mathrm{tr}[\varsigma_\Omega' \varsigma_\Omega \partial_{\Omega\Omega}\pi] + \varsigma_x' \varsigma_\Omega \partial_{\Omega x}\pi \tag{A.26}$$

Substitute μ_x from (A.9) and π, $\partial_x \pi$, $\partial_{xx}\pi$ from (A.24) into equation (A.26) to obtain

$$\begin{aligned} e^{-x}\frac{\beta}{\bar{Q}}\Big[\bar{\iota} + \Phi(x,\pi) - \pi - \varsigma_x' \varsigma_\Omega \partial_\Omega \log \bar{Q} + \mu_\Omega \cdot \partial_\Omega \log \bar{Q} - \frac{1}{\bar{Q}^2}\mathrm{tr}[\varsigma_\Omega' \varsigma_\Omega \partial_{\Omega\Omega}\bar{Q}]\Big] - \kappa f(x) \\ = -\rho\bar{\pi} + \rho(\rho+\beta) + \mu_\Omega \cdot \partial_\Omega \bar{\pi} + \frac{1}{2}\mathrm{tr}[\varsigma_\Omega' \varsigma_\Omega \partial_{\Omega\Omega}\bar{\pi}] \end{aligned} \tag{A.27}$$

The remaining case to consider is if $s_t \to \bar{s}$ constant, in which case all terms involving derivatives with respect to Ω vanish in (A.27), and so

$$e^{-x}\frac{\beta}{\bar{Q}}\Big[\bar{\iota} + \Phi(x,\pi) - \pi\Big] - \kappa f(x) = -\rho\bar{\pi} + \rho(\rho+\beta) \tag{A.28}$$

But equation (A.28) cannot be consistent with the solution for π in (A.24) unless the monetary policy rule Φ takes a knife-edge form, and so generically we reach a contradiction. Thus, $\sigma_x = 0$ must hold. □

Proof of Theorem 3. The strategy of the proof is as follows. First, we characterize the general way in which the present-value Ψ absorbs demand shocks to x, i.e., how the function for Ψ must look for such absorption to occur. Then, we show that the dynamic valuation equation for Ψ can only be consistent with the required function if inflation π takes a particular functional form. Finally, we show that the required inflation function cannot be consistent with the Phillips curve. We need to prove the result for the rigid-price limit ($\kappa \to 0$) and the no-surplus-shock limit ($\varsigma_s \to 0$). We proceed in a general way that nests both cases and then specialize at the end of the proof.

Use Lemma A.1 and x-Markov condition. We specialize the result of Lemma A.1 as follows. First, we have instantaneously-maturing debt, which is nested in the formulas by setting $Q_t = 1$. This implies $\sigma_{Q,t} = 0$ and $\varsigma_{Q,t} = 0$. Second, we have log utility, which implies equation (34) holds, which is the same as equation (A.12) with $\gamma = 1$. Hence, equations (A.16)-(A.17) imply $\Psi_t \sigma_{x,t} = -\sigma_{\Psi,t}$ and $\Psi_t \varsigma_{x,t} = -\varsigma_{\Psi,t}$.

Next, we use the x-Markov assumption (Definition 3) given in the theorem. In that case, and using the surplus dynamics in (33), we have that $\Psi_t = \Psi(x_t, s_t)$ for some function Ψ, as argued in the text. By Itô's formula, this implies that (after dropping t subscripts)

$$\begin{aligned}
\sigma_\Psi &= \sigma_x \partial_x \Psi \\
\varsigma_\Psi &= \varsigma_x \partial_x \Psi + \varsigma_s \partial_s \Psi \\
\mu_\Psi &= \mu_x \partial_x \Psi + \mu_s \partial_s \Psi + \frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2)\partial_{xx}\Psi + \frac{1}{2}\varsigma_s^2 \partial_{ss}\Psi + \varsigma_x \varsigma_s \partial_{sx}\Psi,
\end{aligned}$$

where the surplus drift μ_s is given in (33) by

$$\mu_s = \theta_x x + \theta_\pi \pi + \theta_s(\bar{s} - s) \tag{A.29}$$

Combining these three expressions with equations (A.16)-(A.17) and (A.15), we obtain

$$-\Psi\sigma_x = \sigma_x \partial_x \Psi \tag{A.30}$$

$$-\Psi\varsigma_x = \varsigma_x \partial_x \Psi + \varsigma_s \partial_s \Psi \tag{A.31}$$

$$\rho\Psi - s = \mu_x \partial_x \Psi + \mu_s \partial_s \Psi + \frac{1}{2}(\sigma_x^2 + \varsigma_x^2)\partial_{xx}\Psi + \frac{1}{2}\varsigma_s^2 \partial_{ss}\Psi + \varsigma_x \varsigma_s \partial_{sx}\Psi \tag{A.32}$$

If $\sigma_x \neq 0$, then π is a particular function. Let us assume, leading to contradiction, that $\sigma_x \neq 0$. For (A.30) to hold given the assumption that $\sigma_x = 0$, it must be that $\partial_x \Psi = -\Psi$, and so

$$\Psi(x, s) = \bar{\Psi}(s)e^{-x}$$

for some function $\bar{\Psi}(s)$. Substituting this result into (A.31), it must be that $\varsigma_s \partial_s \Psi = 0$, which means that $\bar{\Psi}(s) = \bar{\Psi}$ constant. Substituting these results into equation (A.32), and also substituting μ_x from (A.9) with $\gamma = 1$, we have

$$e^x s = \big(\bar{\imath} + \Phi(x, \pi) - \pi\big)\bar{\Psi}. \tag{A.33}$$

There are two cases. If $\Phi(x, \pi) - \pi$ is independent of π, then equation (A.33) cannot hold for all (x, s), meaning we have a contradiction, and we are done. If $\Phi(x, \pi) - \pi$ depends on π, then equation (A.33) can only hold if inflation π is a particular function of (x, s). Let us denote this implicit solution by $\tilde{\pi}(x, s)$.

Case (i): rigid-price limit. Note that if $\kappa \to 0$ (rigid price limit), we require $\pi = 0$, so (A.33) cannot hold for all (x,s), which is a contradiction implying $\sigma_x = 0$ above. This proves the claim for the $\kappa \to 0$ case. (Note also that this case does not require the Feller-continuity assumption, nor any assumption on the Taylor rule.)

Case (ii): no-surplus-shock limit. For the no-surplus-shock limit ($\varsigma_s \to 0$ and $\varsigma_x \to 0$), we must continue the line of argument, and we will require the assumed Taylor rule and the Feller continuity condition. Because this part of the proof is slightly more involved, we break it up into steps.

Step 0: notation and proof outline. Let $\mathcal{V}$ denote the region of the state space with volatility, and let $\mathcal{V}^c$ be its complement. Our equilibrium volatility in this case will take the form

$$\sigma_x(x,s)^2 = \begin{cases} \tilde{\sigma}_x(x,s)^2, & \text{if } (x,s) \in \mathcal{V}; \\ 0, & \text{if } (x,s) \in \mathcal{V}^c \end{cases} \tag{A.34}$$

for some function $\tilde{\sigma}_x(x,s)^2$ which is strictly positive on $\mathcal{V}$. The proof consists of the following steps:

1. Identify the unique value of $\tilde{\sigma}_x^2(x,s)$ given a point $(x,s) \in \mathcal{V}$
2. Using this solution, prove that $\tilde{\sigma}_x^2(x,s)$ eventually becomes negative, and so $\mathcal{V}^c$ is non-empty
3. Characterize the equilibrium on the set $\mathcal{V}^c$, in particular the present-value Ψ
4. Conclude by showing Ψ, when pieced together across $\mathcal{V}$ and $\mathcal{V}^c$, violates the Feller-continuity condition, and so $\mathcal{V}^c$ must be the entire space $\mathbb{R}^2$

Step 1: a unique solution for σ_x^2. We continue analyzing the situation $\sigma_x \neq 0$ from our general characterization above. With the linear Taylor rule, the solution to (A.33) is

$$\tilde{\pi}(x,s) = \frac{e^x s \bar{\Psi}^{-1} - \bar{\imath} - \phi_x x}{\phi_\pi - 1} \tag{A.35}$$

Apply Itô's formula to $\pi(x,s)$ and equate the drift to (A.10), denoting $f(x) := \frac{e^{(1+\varphi)x}-1}{1+\varphi}$,

$$\rho\pi - \kappa f(x) = \mu_x \partial_x \pi + \mu_s \partial_s \pi + \frac{1}{2}\sigma_x^2 \partial_{xx}\pi \tag{A.36}$$

Into equation (A.36) we plug μ_x from (A.9), μ_s from (A.29), and the solution $\tilde{\pi}$ from (A.35). This converts (A.36) into an equation for σ_x^2. In particular, define

$$\tilde{\sigma}_x^2 := \Big(e^x s - \frac{1}{2}\bar{\Psi}\phi_x\Big)^{-1}\Big[\rho(e^x s - \bar{\Psi}\bar{\imath} - \bar{\Psi}\phi_x x) - \bar{\Psi}(\phi_\pi - 1)\kappa f(x) \\ - \big[e^x s\bar{\Psi}^{-1} - \rho\big](e^x s - \bar{\Psi}\phi_x) - \big[\theta_x x + \theta_\pi \frac{e^x s\bar{\Psi}^{-1} - \bar{\imath} - \phi_x x}{\phi_\pi - 1} + \theta_s(\bar{s} - s)\big]e^x\Big] \tag{A.37}$$

Then, $\tilde{\sigma}_x^2$ uniquely solves equation (A.36) given $\pi = \tilde{\pi}$. This shows that σ_x^2 must take a particular functional form, i.e., equal to $\tilde{\sigma}_x^2$, whenever it is non-zero, showing that the construction (A.34) is maximally general for x-Markov equilibria.

Step 2: $\tilde{\sigma}_x^2 < 0$ occurs, and so $\mathcal{V}^c$ is non-empty. However, we show next that $\tilde{\sigma}_x^2$ necessarily becomes negative for some values of (x, s). Suppose $\phi_x \neq 0$. Then, if $x \to -\infty$, we have that $f(x) \to -\frac{1}{1+\varphi}$ and so

$$\lim_{x\to-\infty} \tilde{\sigma}_x^2 = \lim_{x\to-\infty} \frac{2}{\phi_x}\Big[\rho(\bar{\imath} + \phi_x x) + \kappa\frac{1-\phi_\pi}{1+\varphi} + \rho\phi_x\Big] = -\infty$$

On the other hand, if $\phi_x = 0$, then take the limit $x \to -\infty$ on the half-plane $s > 0$ to get

$$\lim_{x\to-\infty} \tilde{\sigma}_x^2 = \lim_{x\to-\infty} \frac{1}{s}\Big[\rho(s - e^{-x}\bar{\Psi}\bar{\imath}) - \bar{\Psi}\kappa\frac{1-\phi_\pi}{1+\varphi} + \rho s - \big[\theta_x x + \frac{\theta_\pi \bar{\imath}}{1-\phi_\pi} + \theta_x(\bar{s} - s)\big]\Big] = -\infty,$$

since $\bar{\Psi} > 0$ if the government has positive debt. In either case, we have that $\tilde{\sigma}_x^2$ in (A.37) becomes negative at some finite levels of (x, s).

Define the non-empty set $\mathcal{V}_- := \{(x, s) : \tilde{\sigma}_x^2 < 0\}$. Since $\sigma_x^2 \geq 0$ is required (variance must be positive), we have that the non-volatile set $\mathcal{V}^c$ must contain $\mathcal{V}_-$. Hence, $\mathcal{V}^c$ is non-empty.

Step 3: Characterize equilibrium on the non-volatile set. Next, consider the equilibrium on the set $\mathcal{V}^c$, where $\sigma_x = 0$. Since we are operating in the no-surplus-shock limit, the dynamics of (x, π) are deterministic on this set. Going back to equation (A.36) and plugging μ_x from (A.9), μ_s from (A.29), the assumed Taylor rule, and $\sigma_x = 0$, we have the following PDE for π:

$$\rho\pi - \kappa f(x) = [\phi_x x + (\phi_\pi - 1)\pi]\partial_x \pi + [\theta_x x + \theta_\pi \pi + \theta_s(\bar{s} - s)]\partial_s \pi \quad \text{on } \mathcal{V}^c \tag{A.38}$$

This is a first-order quasilinear PDE with locally Lipschitz continuous coefficients. Hence, by the method of characteristics, a unique solution exists up to a boundary condition.

Let π_0 denote any solution to (A.38).

Going back to equation (A.32), and substituting the expressions for μ_x, μ_s, as well as $\sigma_x = 0$, $\varsigma_x = \varsigma_s = 0$, the Taylor rule, and $\pi = \pi_0$, we obtain a linear differential equation for $\Psi(x, s)$:

$$\rho\Psi - s = [\phi_x x + (\phi_\pi - 1)\pi_0]\partial_x\Psi + [\theta_x x + \theta_\pi\pi_0 + \theta_s(\bar{s} - s)]\partial_s\Psi \quad \text{on } \mathcal{V}^{\mathrm{c}} \tag{A.39}$$

Let Ψ_0 denote the solution to (A.39).

We now prove that Ψ_0 must depend on s. Indeed, suppose not, leading to contradiction. Then, from (A.39), after setting $\partial_s\Psi = 0$, we have

$$\begin{aligned}
\pi_0 &= \frac{s + \phi_x x\partial_x\Psi_0 - \rho\Psi_0}{1 - \phi_\pi} \\
\partial_s\pi_0 &= \frac{1}{1 - \phi_\pi} \\
\partial_x\pi_0 &= \frac{(\phi_x - \rho)\partial_x\Psi_0 + \phi_x x\partial_{xx}\Psi_0}{1 - \phi_\pi}
\end{aligned}$$

Plug these back into (A.38) to obtain

$$\begin{aligned}
0 = &-\rho[s + \phi_x x\partial_x\Psi_0 - \rho\Psi_0] + (1 - \phi_\pi)\kappa f(x) \\
&- [s + \phi_x x\partial_x\Psi_0 - \rho\Psi_0 - \phi_x x]((\phi_x - \rho)\partial_x\Psi_0 + \phi_x x\partial_{xx}\Psi_0) \\
&+ \theta_x x + \frac{\theta_\pi}{1 - \phi_\pi}(s + \phi_x x\partial_x\Psi_0 - \rho\Psi_0) + \theta_s(\bar{s} - s)
\end{aligned}$$

For this to hold for all s and x, given Ψ_0 does not depend on s, the coefficients multiplying s must sum to zero, and then the remaining terms must also sum to zero, i.e., the following two equations both need to hold:

$$0 = -\rho - ((\phi_x - \rho)\partial_x\Psi_0 + \phi_x x\partial_{xx}\Psi_0) + \frac{\theta_\pi}{1 - \phi_\pi} - \theta_s \tag{A.40}$$

$$\begin{aligned}
0 = &-\rho[\phi_x x\partial_x\Psi_0 - \rho\Psi_0] + (1 - \phi_\pi)\kappa f(x) \\
&- [\phi_x x\partial_x\Psi_0 - \rho\Psi_0 - \phi_x x]((\phi_x - \rho)\partial_x\Psi_0 + \phi_x x\partial_{xx}\Psi_0) \\
&+ \theta_x x + \frac{\theta_\pi}{1 - \phi_\pi}(\phi_x x\partial_x\Psi_0 - \rho\Psi_0) + \theta_s\bar{s}
\end{aligned} \tag{A.41}$$

Combine these two equations by eliminating the second derivative, and simplify to obtain

$$0 = (1-\phi_\pi)\kappa f(x) + \Big[\Big(\frac{\theta_\pi}{1-\phi_\pi} - \theta_s - \rho\Big)\phi_x + \theta_x\Big]x + \theta_s(\phi_x x \partial_x \Psi_0 - \rho\Psi_0) + \theta_s \bar{s} \quad \text{(A.42)}$$

As linear ODEs, (A.40) and (A.42) each feature a unique solution. But these solutions need to coincide, which cannot generically be the case. Thus, we have a contradiction, and Ψ_0 cannot be independent of s.

Step 4: Piece together the solution and apply Feller-continuity. Consequently, using these results in conjunction with our previous results on the volatile set $\mathcal{V}$, we have that

$$\Psi(x,s) = \begin{cases} \bar{\Psi}e^{-x}, & \text{if } (x,s) \in \mathcal{V}; \\ \Psi_0(x,s), & \text{if } (x,s) \in \mathcal{V}^c. \end{cases} \quad \text{(A.43)}$$

In particular, since Ψ_0 must depend on s by the previous step, we have that Ψ cannot be continuous on the boundary $\partial\mathcal{V}$ of the volatile set.

Recall the definition of the present-value Ψ in (A.12), which implies in our current x-Markov equilibrium that

$$\Psi(x_t, s_t) = \mathbb{E}_t\Big[\int_t^\infty e^{-\rho(u-t)} s_u du\Big]$$

for some function Ψ. Using Feller-continuity, we have that Ψ must be a continuous function, which contradicts equation (A.43). This contradiction proves that either $\mathcal{V}$ is empty of $\mathcal{V}^c$ is empty. But step 2 has already proven that $\mathcal{V}^c$ is non-empty. Hence, $\mathcal{V}$ is empty, implying $\sigma_x = 0$ everywhere. □

Proof of Theorem 4. The strategy of the proof is very similar to Theorem 3.

We specialize the result of Lemma A.1 as follows. First, we have instantaneously-maturing debt, which is nested in the above formulas by setting $Q_t = 1$. This implies $\sigma_{Q,t} = 0$ and $\varsigma_{Q,t} = 0$. Hence, equations (A.16)-(A.17) imply $\gamma\Psi_t\sigma_{x,t} = -\sigma_{\Psi,t}$ and $\gamma\Psi_t\varsigma_{x,t} = -\varsigma_{\Psi,t}$.

Next, we use the x-Markov assumption given in the theorem. In an x-Markov equilibrium, we have that $\sigma_{x,t} = \sigma_x(x_t,\Omega_t)$, $\varsigma_{x,t} = \varsigma_x(x_t,\Omega_t)$, and $\pi_t = \pi(x_t,\Omega_t)$ for some functions σ_x, ς_x, and π. In that case, the nominal interest rate $\iota_t = \bar{\iota} + \Phi(x_t,\pi_t) = \bar{\iota} + \Phi(x_t,\pi(x_t,\Omega_t))$ is purely a function of (x_t,Ω_t). These facts imply furthermore that (x_t,Ω_t) is a Markov diffusion (i.e., their dynamics only depend on x_t and Ω_t). Since $s_t = s(\Omega_t)$ and $Y_t = Y^* e^{x_t}$, equation (A.12) for Ψ_t implies that there exists some function

Ψ such that $\Psi_t = \Psi(x_t, \Omega_t)$.

Apply Itô's formula to Ψ to obtain (after dropping t subscripts)

$$\begin{aligned}
\sigma_\Psi &= \sigma_x \partial_x \Psi \\
\varsigma_\Psi &= \varsigma_x \partial_x \Psi + \varsigma_\Omega \partial_\Omega \Psi \\
\mu_\Psi &= \mu_x \partial_x \Psi + \mu_\Omega' \partial_\Omega \Psi + \frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2)\partial_{xx}\Psi + \frac{1}{2}\mathrm{tr}[\varsigma_\Omega' \varsigma_\Omega \partial_{\Omega\Omega}\Psi] + \varsigma_x' \varsigma_\Omega \partial_{\Omega x}\Psi
\end{aligned}$$

Combining these three expressions with equations (A.16)-(A.17) and (A.15), we obtain

$$-\Psi\gamma\sigma_x = \sigma_x \partial_x \Psi \tag{A.44}$$

$$-\Psi\gamma\varsigma_x = \varsigma_x \partial_x \Psi + \varsigma_\Omega \partial_\Omega \Psi \tag{A.45}$$

$$\begin{aligned}
\rho\Psi - s(Y^*)^{1-\gamma}e^{(1-\gamma)x} = \mu_x \partial_x \Psi + \mu_\Omega' \partial_\Omega \Psi + \frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2)\partial_{xx}\Psi \\
+ \frac{1}{2}\mathrm{tr}[\varsigma_\Omega' \varsigma_\Omega \partial_{\Omega\Omega}\Psi] + \varsigma_x' \varsigma_\Omega \partial_{\Omega x}\Psi
\end{aligned} \tag{A.46}$$

Let us assume, leading to contradiction, that $\sigma_x \neq 0$. For (A.44) to hold given the assumption that $\sigma_x = 0$, it must be that $\partial_x \Psi = -\gamma\Psi$, and so

$$\Psi(x, \Omega) = \bar{\Psi}(\Omega)e^{-\gamma x}$$

Substituting this result into (A.45), it must be that $\varsigma_\Omega \partial_\Omega \Psi = 0$, which means that $\bar{\Psi}(\Omega) = \bar{\Psi}$ constant. Substituting these results into equation (A.46), and also substituting μ_x from (A.9), we have

$$(Y^*)^{1-\gamma}e^x s = \big(\bar{\imath} + \Phi(x, \pi) - \pi\big)\bar{\Psi} \tag{A.47}$$

which can only hold if inflation π is a particular function of (x, Ω). There are two cases. If $\Phi(x, \pi) - \pi$ is independent of π, then equation (A.47) cannot hold for all (x, Ω), meaning we have a contradiction, and we are done. If $\Phi(x, \pi) - \pi$ depends on π, then equation (A.47) can only hold if inflation π is a particular function of (x, Ω). Let us denote this solution by $\tilde{\pi}(x, \Omega)$.

Note that if $\kappa \to 0$ (rigid price limit), we have $\pi = 0$, so (A.47) cannot hold for all (x, Ω), which is a contradiction implying $\sigma_x = 0$ above. This proves the claim for the $\kappa \to 0$ case. (Note also that this case does not require the assumption on the Taylor rule nor the clause about the "long-run equilibrium".)

We next need to address the constant surplus-output ratio case $s_t = \bar{s}$, which requires the assumed Taylor rule. With the linear Taylor rule and $s = \bar{s}$, the solution to (A.47) is

$$\tilde{\pi}(x) = \frac{Ge^x - \bar{\iota} - \phi_x x}{\phi_\pi - 1}, \quad \text{where} \quad G := (Y^*)^{1-\gamma}\bar{\Psi}^{-1}\bar{s} > 0 \tag{A.48}$$

Apply Itô's formula to $\pi(x)$, and equate the drift to (A.10), denoting $f(x) := \frac{e^{(\gamma+\varphi)x}-1}{\gamma+\varphi}$,

$$\rho\pi - \kappa f(x) = \mu_x \partial_x \pi + \frac{1}{2}\sigma_x^2 \partial_{xx}\pi \tag{A.49}$$

Into equation (A.49) we plug μ_x from (A.9) and the solution $\tilde{\pi}$ from (A.48). This converts (A.49) into an equation for σ_x^2. In particular, define

$$\tilde{\sigma}_x^2 := \frac{2}{\gamma}\frac{\gamma\rho(Ge^x - \bar{\iota} - \phi_x x) - \gamma(\phi_\pi - 1)\kappa f(x) - (Ge^x - \rho)(Ge^x - \phi_x)}{(\gamma+1)Ge^x - \gamma\phi_x} \tag{A.50}$$

Then, $\tilde{\sigma}_x^2$ solves equation (A.49) given $\pi = \tilde{\pi}$. This shows that σ_x^2 must take a particular functional form, i.e., equal to $\tilde{\sigma}_x^2$, whenever it is non-zero.

However, we show next that $\tilde{\sigma}_x^2$ necessarily becomes negative for some values of x. Suppose $\phi_x \neq 0$. Then, if $x \to -\infty$, we have that $f(x) \to -\frac{1}{\gamma+\varphi}$ and so

$$\lim_{x\to-\infty} \tilde{\sigma}_x^2 = \lim_{x\to-\infty} \frac{2}{\gamma}\frac{\gamma\rho(\bar{\iota} + \phi_x x) + \gamma\kappa\frac{1-\phi_\pi}{\gamma+\varphi} + \rho\phi_x}{\gamma\phi_x} = -\infty$$

On the other hand, if $\phi_x = 0$, then

$$\lim_{x\to-\infty} \tilde{\sigma}_x^2 = \lim_{x\to-\infty} \frac{2}{\gamma}\frac{\gamma\rho G - \gamma\rho\bar{\iota}e^{-x} - \gamma\kappa\frac{1-\phi_\pi}{\gamma+\varphi}e^{-x} + \rho G}{(\gamma+1)G} = -\infty,$$

since $G > 0$ if the government has positive debt (i.e., $\bar{s} > 0$ and $\bar{\Psi} > 0$). In either case, we have that $\tilde{\sigma}_x^2$ in (A.37) becomes negative at some finite level of x. On the other hand, substituting $x = 0$ in (A.50), and using the target rate $\bar{\iota} = \rho$, we have

$$\tilde{\sigma}_x^2(0) = \frac{2}{\gamma}(G - \rho)\frac{\gamma\rho + \phi_x - G}{(\gamma+1)G - \gamma\phi_x}$$

Note that $G > \gamma\rho + \phi_x$ if $\gamma > 1$ and ε is not too large. To see this, recall that $Y^* = (\frac{\varepsilon-1}{\varepsilon})^{\frac{1}{\gamma+\varphi}}$, so that small enough ε can make Y^* arbitrarily small. Hence, $G = (Y^*)^{1-\gamma}\bar{\Psi}^{-1}\bar{s}$ becomes arbitrarily large for small ε. Therefore, $\tilde{\sigma}_x^2(0)$ necessarily becomes negative for small enough ε and $\gamma > 1$.

Thus, we have proven that the set $\mathcal{V}_- := \{x : \tilde{\sigma}_x^2 \leq 0\}$ is non-empty. Let $\mathcal{V}$ be any subset of $\mathbb{R}\backslash\mathcal{V}_-$. Put

$$\sigma_x^2 = \begin{cases} \tilde{\sigma}_x^2, & \text{if } x \in \mathcal{V}; \\ 0, & \text{if } x \in \mathcal{V}^c. \end{cases} \tag{A.51}$$

Every x-Markov equilibrium that satisfies (A.48)-(A.49) and has $\sigma_x \neq 0$ must take the form (A.51) for some $\mathcal{V} \subset \mathbb{R}^2$, because we cannot have $\sigma_x^2 < 0$. However, since our approach will evaluate the state dynamics on the boundary $\partial\mathcal{V}$, it suffices to assume this boundary is minimal (i.e., two points). In particular, given $\tilde{\sigma}^2(-\infty) < 0$ and $\tilde{\sigma}^2(0) < 0$, we assume without loss of generality that $\mathcal{V}^c = (\underline{x}, \overline{x})$ is an interval in the negative real line, i.e., $\underline{x} < \overline{x} < 0$.

For x_t to have volatility in its long-run distribution, it must be the case that its dynamics keep it in the interval $(\underline{x}, \overline{x})$. Since the function $\tilde{\sigma}_x^2$ is continuous, it must be bounded at $\underline{x}$, so it cannot cause the drift μ_x to diverge. In that case, the only way $x_t \geq \underline{x}$ can hold with probability 1 is if

$$\sigma_x^2(\underline{x}) = 0 \quad \text{and} \quad \mu_x(\underline{x}) = \gamma^{-1}[\phi_x \underline{x} + (\phi_\pi - 1)\tilde{\pi}(\underline{x})] \geq 0$$

On the other hand, the only way $x_t \leq \overline{x}$ can hold with probability 1 is if

$$\sigma_x^2(\overline{x}) = 0 \quad \text{and} \quad \mu_x(\overline{x}) = \gamma^{-1}[\phi_x \overline{x} + (\phi_\pi - 1)\tilde{\pi}(\overline{x})] \leq 0$$

Substituting $\tilde{\pi}$ from (A.48), we have $\phi_x x + (\phi_\pi - 1)\tilde{\pi}(x) = Ge^x - \bar{\iota}$, and so the boundary requirements are

$$0 \leq Ge^{\underline{x}} - \bar{\iota}$$
$$0 \geq Ge^{\overline{x}} - \bar{\iota}$$

which cannot both hold since $G > 0$ and $\overline{x} > \underline{x}$. This contradiction implies that $\sigma_{x,t} = 0$ in the long-run distribution. $\square$

A.5 Proof of Proposition 6

First, note that the spectral decomposition of $\mathcal{A} = V\Lambda V^{-1}$ is

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} v(\lambda_1) & v(\lambda_2) \end{bmatrix},$$

where the eigenvalues λ_1, λ_2 and the corresponding eigenvectors $v(\lambda_1), v(\lambda_2)$ are

$$\begin{aligned} \lambda_1 &= \frac{1}{2}\left[\rho + \phi_x + \sqrt{(\rho - \phi_x)^2 - 4\kappa(\phi_\pi - 1)}\right] \\ \lambda_2 &= \frac{1}{2}\left[\rho + \phi_x - \sqrt{(\rho - \phi_x)^2 - 4\kappa(\phi_\pi - 1)}\right] \\ \text{and} \quad v(\lambda) &= \begin{pmatrix} \frac{\phi_\pi - 1}{\lambda - \phi_x} \\ 1 \end{pmatrix}. \end{aligned}$$

Recall equation (37) that

$$\mathbb{E}_0[\tilde{F}_t] = \exp(\Lambda t)\tilde{F}_0, \tag{A.52}$$

where $\tilde{F}_t = V^{-1}F_t$ is a rotated version of the state $F_t = (x_t, \pi_t)'$, and where

$$V^{-1} = \frac{1}{\lambda_2 - \lambda_1}\begin{bmatrix} \frac{(\lambda_1 - \phi_x)(\lambda_2 - \phi_x)}{\phi_\pi - 1} & -(\lambda_1 - \phi_x) \\ -\frac{(\lambda_1 - \phi_x)(\lambda_2 - \phi_x)}{\phi_\pi - 1} & \lambda_2 - \phi_x \end{bmatrix}.$$

In equation (A.52), $\exp(\Lambda t)$ refers to element-by-element exponentiation of Λ.

Let's consider the three cases of the proposition, using Condition 1 to kill explosive solutions to (A.52):

1. *Case 1:* $\rho + \phi_x > 0$ and $\rho\phi_x + \kappa(\phi_\pi - 1) > 0$.

   In this case, $\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2) > 0$. Therefore, the only non-explosive solution to (A.52) is $\tilde{F}_t = 0$, which implies $F_t = 0$, i.e., $x_t = \pi_t = 0$.

2. *Case 2:* $\rho\phi_x + \kappa(\phi_\pi - 1) < 0$.

   In this case, both eigenvalues are real and have opposite signs: $\lambda_1 > 0 > \lambda_2$. Therefore, all non-explosive solutions to (A.52) must satisfy $\tilde{F}_t^{(1)} = 0$, which using the expression for V^{-1} implies

   $$\pi_t = \frac{\lambda_2 - \phi_x}{\phi_\pi - 1}x_t.$$

Given $\sigma_{x,t} = 0$ from Theorem 1, this then implies $\sigma_{\pi,t} = 0$ as well.

3. *Case 3:* $\rho + \phi_x < 0$ and $\rho\phi_x + \kappa(\phi_\pi - 1) > 0$.

   In this case, $\text{Re}(\lambda_1), \text{Re}(\lambda_2) < 0$, meaning all initial conditions to (A.52) are non-explosive. Therefore, any $\tilde{F}_0$ corresponds to a valid equilibrium.

In all cases, we note that x_0 and $\varsigma_{x,t}$ are pinned down by (GD) at $t = 0$ and at $t > 0$, respectively. Indeed, using $ds_t = \lambda_s[s_t - \bar{s}]dt + \varsigma_{s,t} \cdot d\mathcal{Z}_t$ in equation (A.12), we obtain

$$\Psi_t = \frac{\bar{s}}{\rho} + \frac{s_t - \bar{s}}{\rho + \lambda_s},$$

which is exogenous. Using Ψ_0 in the $t = 0$ version of (GD), we obtain

$$x_0 = \log\left(\frac{B_0}{\Psi_0 P_0 Y^*}\right). \tag{A.53}$$

On the other hand, for $t > 0$, we have $\varsigma_\Psi = \frac{1}{\rho+\lambda_s}\varsigma_s$. Apply this in equation (A.17) of Lemma A.1, with $\gamma = 1$ and $Q \equiv 1$, to obtain

$$\varsigma_{x,t} = -\frac{\rho}{\lambda_s \bar{s} + \rho s_t}\varsigma_{s,t}. \tag{A.54}$$

Thus, x_0 and $\varsigma_{x,t}$ are pinned down.

The remaining claims to prove are the existence/uniqueness statements. In Case 1, the equilibrium fails to exist generically, because $x_t = 0$ cannot be consistent with (A.53) and (A.54). In Case 2, the equilibrium is unique, because the initial conditions x_0 and $\pi_0 = \frac{\lambda_2 - \phi_x}{\phi_\pi - 1}x_0$ are pinned down by (A.53), and because the surplus shock exposures $\varsigma_{x,t}$ and $\varsigma_{\pi,t} = \frac{\lambda_2 - \phi_x}{\phi_\pi - 1}\varsigma_{x,t}$ are pinned down by (A.54). In Case 3, the equilibrium is not unique because, although x_0 is pinned down by (A.53), π_0 is not pinned down. Furthermore, π_t can have arbitrary sunspot volatility $\sigma_{\pi,t}$, despite the fact that $\sigma_{x,t} = 0$.

B Inflation Dynamics under Rotemberg

Here, we generalize the sticky-price model of Rotemberg (1982) to our environment. Since firms in our economy are ex-ante identical, they will have identical utilization and price-setting incentives, allowing us to study a representative firm's problem and a symmetric equilibrium.

To set up the representative intermediate-goods-producer problem, let l_t denote the firm's hired labor, at some equilibrium wage W_t. The firm produces $y_t = l_t$. The firm makes its price choice p_t, internalizing its demand $y_t = (p_t/P_t)^{-\varepsilon} Y_t$, where P_t and Y_t are the aggregate price and output. This demand curve comes from an underlying Dixit-Stiglitz structure with CES preferences (with substitution elasticity $\varepsilon > 1$) and monopolistic competition in the intermediate goods sector.

Letting M_t denote the real SDF process, the representative firm solves

$$\sup_{p,l} \mathbb{E}\Big[\int_0^\infty M_t\Big(\frac{p_t}{P_t}y_t - \frac{W_t l_t}{P_t} - \frac{1}{2\eta}\big(\frac{1}{dt}\frac{dp_t}{p_t}\big)^2 Y_t\Big)dt\Big] \tag{B.1}$$

$$\text{subject to} \quad y_t = (p_t/P_t)^{-\varepsilon} Y_t \tag{B.2}$$

$$y_t = l_t \tag{B.3}$$

The quadratic price adjustment cost in (B.1) has a penalty parameter η. As $\eta \to 0$ ($\eta \to \infty$), prices become permanently rigid (flexible). We assume that this price adjustment cost is purely non-pecuniary for simplicity (this means that adjustment costs do not affect the resource constraint). Alternatively, we could redistribute these adjustment costs lump-sum to the representative household.

Before solving the problem, we can immediately note the following property: price changes are necessarily absolutely continuous ("order dt"). Indeed, the adjustment cost per unit of time is a function of price changes per unit of time, i.e., $\frac{1}{dt}\frac{dp_t}{p_t}$. If prices were to change faster than dt, say with the Brownian motion dZ_t, then $\frac{1}{dt}\frac{dp_t}{p_t}$ would be unbounded almost-surely (because Brownian motion is nowhere-differentiable), leading to infinite adjustment costs. Consequently, we know that $\frac{1}{dt}\frac{dp_t}{p_t} = \frac{\dot{p}_t}{p_t}$ for some $\dot{p}_t$.

The firm's optimal price sequence solves a dynamic optimization problem. Substituting the demand curve from (B.2) and the production function from (B.3), we may rewrite problem (B.1) as

$$\sup_{\dot{p}} \mathbb{E}_t\Big[\int_t^\infty \frac{M_s Y_s}{M_t Y_t}\Big(\big(\frac{p_s}{P_s}\big)^{1-\varepsilon} - \frac{W_s}{P_s}\big(\frac{p_s}{P_s}\big)^{-\varepsilon} - \frac{1}{2\eta}\big(\frac{\dot{p}_s}{p_s}\big)^2\Big)ds\Big].$$

Furthermore, note that in the log utility model used in the text, we have $M_t Y_t = e^{-\rho t}$. Letting J denote this firm's value function, the HJB equation is

$$0 = \sup_{\dot{p}_t}\left\{\big(\frac{p_t}{P_t}\big)^{1-\varepsilon} - \frac{W_t}{P_t}\big(\frac{p_t}{P_t}\big)^{-\varepsilon} - \frac{1}{2\eta}\big(\frac{\dot{p}_t}{p_t}\big)^2 - \rho J_t + \frac{1}{dt}\mathbb{E}_t[dJ_t]\right\}$$

The firm value function follows a process of the form

$$dJ_t = [\mu_{J,t} + \dot{p}_t \frac{\partial}{\partial p} J_t]dt + \sigma_{J,t} dZ_t,$$

where $\mu_{J,t}$ and $\sigma_{J,t}$ are only functions of aggregate states (not the individual price). The only part that the firm can affect is $\dot{p}_t \frac{\partial}{\partial p} J_t$. Plugging these results back into the HJB equation and taking the FOC, we have

$$0 = -\frac{1}{\eta}\big(\frac{\dot{p}_t}{p_t}\big)\frac{1}{p_t} + \frac{\partial}{\partial p} J_t \tag{B.4}$$

Differentiating the HJB equation with respect to the state variable p_t, we have the envelope condition

$$(\varepsilon - 1)\big(\frac{p_t}{P_t}\big)^{-\varepsilon}\frac{1}{P_t} - \varepsilon\frac{W_t}{P_t}\big(\frac{p_t}{P_t}\big)^{-\varepsilon-1}\frac{1}{P_t} = \frac{1}{\eta}\big(\frac{\dot{p}_t}{p_t}\big)^2\frac{1}{p_t} - \rho\frac{\partial}{\partial p}J_t + \frac{1}{dt}\mathbb{E}_t\big[d\big(\frac{\partial}{\partial p}J_t\big)\big], \tag{B.5}$$

where the last term uses the stochastic Fubini theorem. Combining equations (B.4) and (B.5), we have

$$\eta(\varepsilon - 1)\big(\frac{p_t}{P_t}\big)^{-\varepsilon}\frac{1}{P_t} - \eta\varepsilon\frac{W_t}{P_t}\big(\frac{p_t}{P_t}\big)^{-\varepsilon-1}\frac{1}{P_t} = \big(\frac{\dot{p}_t}{p_t}\big)^2\frac{1}{p_t} - \rho\big(\frac{\dot{p}_t}{p_t}\big)\frac{1}{p_t} + \frac{1}{dt}\mathbb{E}_t\big[d\big(\big(\frac{\dot{p}_t}{p_t}\big)\frac{1}{p_t}\big)\big] \tag{B.6}$$

At this point, define the firm-level inflation rate $\pi_t := \dot{p}_t/p_t$, note that $\mathbb{E}_t\big[d\big(\pi_t\frac{1}{p_t}\big)\big] = \frac{1}{p_t}\mathbb{E}_t[d\pi_t] - \frac{1}{p_t}\pi_t^2 dt$, and use the symmetry assumption $p_t = P_t$ in (B.6) to get

$$\eta(\varepsilon - 1) - \eta\varepsilon\frac{W_t}{P_t} = -\rho\pi_t + \frac{1}{dt}\mathbb{E}_t[d\pi_t]. \tag{B.7}$$

Equation (B.7) is the continuous-time stochastic Phillips curve, with π_t interpreted also as the aggregate inflation rate (given a symmetric equilibrium).

Finally, note that the firm's optimization problem also requires the following transversality condition (see Theorem 9.1 of Fleming and Soner (2006)):

$$\lim_{T\to\infty}\mathbb{E}_t[M_T Y_T J_T] = 0.$$

In a symmetric equilibrium ($p = P$), and using the log utility result $M_t Y_t = e^{-\rho t}$, we have that

$$M_T Y_T J_T = \mathbb{E}_T\Big[\int_T^\infty e^{-\rho t}\Big(1-(Y^*)^{1+\varphi}e^{(1+\varphi)x_t}-\frac{1}{2\eta}\pi_t^2\Big)dt\Big]$$

Take expectations and the limit $T \to \infty$. Sufficient conditions for the result to be zero are

$$\lim_{T\to\infty}\mathbb{E}_t[e^{(1+\varphi)x_T-\rho T}] = 0 \tag{B.8}$$

$$\lim_{T\to\infty}\mathbb{E}_t[e^{-\rho T}\pi_T^2] = 0 \tag{B.9}$$

Equation (B.8) is identical to the one of the requirements for the consumer's problem to be well-defined (see Appendix A.1). Equation (B.9) avoids nominal explosions that imply an infinite present value of adjustment costs. Note that under Condition 1, both of these equations automatically hold.

C Nonlinear Phillips Curve

This section briefly explores the stability properties of the nonlinear Phillips curve, in contrast the linearized version used oftentimes in the paper. We will do this only in the context of deterministic equilibria, for simplicity. For convenience, we repeat this nonlinear equation here:

$$\dot{\pi}_t = \rho\pi_t - \kappa\Big(\frac{e^{(1+\varphi)x_t}-1}{1+\varphi}\Big). \tag{C.1}$$

We also repeat the IS curve after substituting the linear Taylor rule with target rate $\bar{\imath} = \rho$:

$$\dot{x}_t = \phi_x x_t + (\phi_\pi - 1)\pi_t. \tag{C.2}$$

A deterministic non-explosive equilibrium in this environment is (x_t, π_t) that satisfy (C.1)-(C.2) and asymptotic non-explosion Condition 1.

The nonlinearity of the Phillips curve does not change the basic determinacy result of Proposition 1, as we show next (although our proof requires stronger assumptions on the Taylor rule to ensure global determinacy).

Proposition C.1. *Consider the system (C.1)-(C.2) with $\phi_x > \rho$ and $\phi_\pi > 1$. Then, the only initial pair (x_0, π_0) consistent with a deterministic non-explosive equilibrium is $(x_0, \pi_0) = (0, 0)$. Any other initial pair diverges, but only asymptotically (i.e., not in finite time).*

PROOF OF PROPOSITION C.1. Define $f(x) := \frac{e^{(1+\varphi)x}-1}{1+\varphi}$. From (C.1)-(C.2), the steady state solves

$$-\phi_x x = (\phi_\pi - 1)\kappa\rho^{-1} f(x)$$

The two sides of this equation have opposite slopes in x, so the unique solution is $x = 0$, proving the unique steady state is $(x, \pi) = (0, 0)$. The steady state is locally unstable, by the same linearized eigenvalue analysis leading to Proposition 1. By the local stable manifold theorem, we have that the unique stable solution to the dynamics is in fact this steady state. We now prove that any non-explosive equilibrium (satisfying Condition 1) must have $(x_t, \pi_t) = (0, 0)$ for all t. Assume not, i.e., assume, leading to contradiction, that $x_t \in [\underline{x}, \overline{x}]$ for all $t > 0$, where $\underline{x} < 0 < \overline{x}$.

First, from (C.1),

$$e^{-\rho t}\pi_t - \pi_0 = -\kappa \int_0^t e^{-\rho s} f(x_s) ds \tag{C.3}$$

Substituting (C.3) into (C.2), we have

$$\dot{x}_t = \phi_x x_t + (\phi_\pi - 1)\left[e^{\phi_x t}\pi_0 - \frac{\kappa}{\rho}\int_0^t \rho e^{\rho(t-u)} f(x_u) du\right] \tag{C.4}$$

Under the boundedness assumption, we may bound $f(\underline{x}) \leq f(x_t) \leq f(\overline{x})$, which when plugging into (C.4) leads to

$$\underbrace{\phi_x x_t + (\phi_\pi - 1)\left[e^{\phi_x t}\pi_0 - \frac{\kappa}{\rho}(e^{\rho t} - 1) f(\overline{x})\right]}_{:=L_t} \leq \dot{x}_t \leq \underbrace{\phi_x x_t + (\phi_\pi - 1)\left[e^{\phi_x t}\pi_0 - \frac{\kappa}{\rho}(e^{\rho t} - 1) f(\underline{x})\right]}_{:=U_t}$$

If $\pi_0 > 0$, then $L_t, U_t \to +\infty$ as $t \to \infty$ for every possible value of $x_t \in [\underline{x}, \overline{x}]$. On the other hand, if $\pi_0 < 0$, then $L_t, U_t \to -\infty$ as $t \to \infty$ for every possible value of $x_t \in [\underline{x}, \overline{x}]$. Hence, $\pi_0 > 0$ implies $x_T > \overline{x}$ for some $T > 0$, while $\pi_0 < 0$ implies $x_T < \underline{x}$ for some $T > 0$. This contradicts the bounded set $x_t \in [\underline{x}, \overline{x}]$, which implies $\pi_0 = 0$ is required.

However, since time 0 is arbitrary in this analysis, and the entire argument could be shifted forward in time, we in fact require $\pi_t = 0$ for all $t \geq 0$. Going back to equation (C.1), we then have that $x_t = 0$ for all $t \geq 0$. □

D Nuclear Taylor Rules and Finite-Time Explosions

Suppose we would like to allow deterministic equilibria that explode asymptotically, in violation of Condition 1. For instance, Cochrane (2011) considers some types of asymptotically exploding equilibria in his argument for non-uniqueness. In that case, is the spirit of Proposition 1 still true, i.e., do there exist Taylor rules which can eliminate indeterminacies? The answer is yes, but a "nuclear Taylor rule" is required to force explosion in finite time.

In particular, let us dispense with the linear rule (linear MP). Suppose the response function (MP) takes the nonlinear form

$$\Phi(x, \pi) = \frac{\phi_x}{2}(e^x - e^{-x}) + \pi \tag{D.1}$$

with $\phi_x > 0$ and suppose the target rate is again the natural rate $\bar{\imath} = \rho$. Note that the log-linearized version of (D.1) renders the linear Taylor rule (linear MP) with $\phi_\pi = 1$.

Combining (D.1) with (IS), the dynamics of x_t are given by

$$\dot{x}_t = \frac{\phi_x}{2}(e^{x_t} - e^{-x_t}) \tag{D.2}$$

This ODE has solution

$$x_t = \log\left(\frac{1 - Ke^{\phi_x t}}{1 + Ke^{\phi_x t}}\right)$$

where $K = \frac{1-e^{x_0}}{1+e^{x_0}}$. This process diverges in *finite time* for any $x_0 \neq 0$: it explodes at time $T = -\phi_x^{-1}\log(|K|)$. Hence, we have proved by construction the following result.

Proposition D.1. *Taylor rules exist such that any deterministic equilibrium has $x_t = 0$ forever.*

The analysis above abstracts from any feedback effects from inflation to output gap by setting a monetary policy rule with $\phi_\pi = 1$. This serves two purposes. First, it emphasizes the focus on self-fulfilling demand and not inflation per se. Equilibrium characterization requires the output gap to remain bounded for any finite horizon. There is no such requirement for inflation (e.g., hyperinflation might be an equilibrium outcome). Second, it simplifies the analysis and illustrates the point with examples that permit closed form solutions. As an additional benefit, Proposition D.1 holds for either the linearized or non-linear Phillips curves.

Determinacy extends beyond the particular response function (D.1) that has exactly a one-for-one inflation response. In particular, consider inflation sensitivities of more than one-for-one, such as

$$\Phi(x, \pi) = \frac{\phi_x}{2}(e^x - e^{-x}) + \phi_\pi \pi, \quad \phi_x > 0, \phi_\pi > 1. \tag{D.3}$$

While more challenging technically to analyze, this rule also selects the zero output gap equilibrium $x_t = 0$. We demonstrate this result formally next.

Under rule (D.3), the dynamical system for (x_t, π_t) is

$$\dot{\pi}_t = \rho \pi_t - \kappa f(x_t) \tag{D.4}$$

$$\dot{x}_t = \frac{\phi_x}{2}(e^{x_t} - e^{-x_t}) + (\phi_\pi - 1)\pi_t \tag{D.5}$$

where $f(x) := (1+\varphi)^{-1}[e^{(1+\varphi)x} - 1]$.

Proposition D.2. *Consider the system (D.4)-(D.5) with $\phi_x > 0$ and $\phi_\pi > 1$. Then, $(x_t, \pi_t) = (0, 0)$ is the unique equilibrium that does not explode in finite time.*

Proof of Proposition D.2. Suppose the solution $(x_t(\phi_\pi), \pi_t(\phi_\pi))_{t\geq 0}$ associated to some $\phi_\pi > 1$ (which is unique prior to an explosion by the standard ODE uniqueness theorem) did not explode in finite time. In that case, because the solution is continuous in ϕ_π (again, standard ODE theorems ensure this), it follows that the solution $(x_t(\tilde{\phi}_\pi), \pi_t(\tilde{\phi}_\pi))_{t\geq 0}$ associated with $\tilde{\phi}_\pi < \phi_\pi$ also does not explode in finite time. Continuity requires this: otherwise, the two solutions would be infinitely far apart at some finite time T when one of the solutions does explode. But Proposition D.1 has already shown that $(x_t(1), \pi_t(1))_{t\geq 0}$ is explosive in finite time, a contradiction. □

E Sunspot equilibria with inflation

In the sunspot equilibrium constructions of Propositions 2-3, we work in the rigid price limit ($\kappa \to 0$) for analytical tractability. Here, we provide one example construction where prices are partially flexible, so inflation is present. For this example, we will assume the linearly approximated Phillips curve (linear PC) and utilize a linear Taylor rule (linear MP) that is sufficiently aggressive. In particular, we will assume $\phi_x > 0$ and $\phi_\pi > 1$, so the Taylor principle is satisfied and deterministic multiplicities (as well as linearized stochastic multiplicities) are ruled out. To obtain an analytical solution, we will also need to assume $(\phi_x - \rho)^2 > 4\kappa(\phi_\pi - 1)$.

To maintain tractability, we assume a type of Markovian equilibrium where inflation is a function of the output gap. In particular, suppose $\pi_t = \pi(x_t)$ for some function $\pi(\cdot)$, to be determined. Obviously, this must be supported by a volatility process σ_x which is solely a function of x. These restrictions imply only one dimension of multiplicity, but the equilibrium can still be relatively rich. By Proposition 3, we need only consider sunspot equilibria with $x \leq 0$. In fact, let us split the space into $(-\infty, \bar{x})$ and $(\bar{x}, \infty)$, for $\bar{x} < 0$, and put $\sigma_x = 0$ for all $x \geq \bar{x}$ and $\sigma_x \neq 0$ for $x < \bar{x}$. Below, we provide one example class of equilibria and prove they are valid. After the statement of the proposition and its proof, we provide a numerical illustration in this class, where we can even relax some of the conditions.

Proposition E.1. *Suppose $\phi_x > 0$, $\phi_\pi > 1$, and $(\phi_x - \rho)^2 > 4\kappa(\phi_\pi - 1)$. Define the following constants:*

$$\begin{aligned}
\bar{x}_{max} &:= \frac{\rho}{\phi_x - (\phi_\pi - 1)\bar{\pi}_1} \frac{\rho - \bar{\imath}}{\rho + (\phi_\pi - 1)\bar{\pi}_1} \\
\bar{\pi}_0 &:= \frac{(\rho - \bar{\imath})\bar{\pi}_1}{\rho + (\phi_\pi - 1)\bar{\pi}_1} \\
\bar{\pi}_1 &:= \frac{1}{2(\phi_\pi - 1)}\left[\phi_x - \rho + \sqrt{(\phi_x - \rho)^2 - 4\kappa(\phi_\pi - 1)}\right]
\end{aligned}$$

Let $(\bar{x}, \hat{\pi}_0, \hat{\pi}_1)$ be any constants satisfying

$$\bar{x} < \min[0, \bar{x}_{max}] \tag{E.1}$$

$$\hat{\pi}_1 > \begin{cases} 0, & \text{if } \hat{\pi}_0 < -\frac{\phi_x^2}{4\rho(\phi_\pi - 1)}; \\ \frac{2\rho\hat{\pi}_0}{-\phi_x + \sqrt{\phi_x^2 + 4\rho\hat{\pi}_0(\phi_\pi - 1)}}, & \text{otherwise.} \end{cases} \tag{E.2}$$

$$0 > \hat{\pi}_0 > -\hat{\pi}_1 - \frac{\lambda\hat{\pi}_1(\bar{\imath} - \rho)}{\rho + \lambda\hat{\pi}_1(\phi_\pi - 1)} \tag{E.3}$$

For any $\lambda \in (1, 2)$, define the function

$$\pi(x) = \begin{cases} \bar{\pi}_0 - \bar{\pi}_1 x, & \text{if } x \geq \bar{x}; \\ \hat{\pi}_0 + \hat{\pi}_1 e^{-\lambda x}, & \text{if } x < \bar{x}. \end{cases} \tag{E.4}$$

Then, an equilibrium exists in which inflation is given by $\pi_t = \pi(x_t)$, volatility by some $\sigma_{x,t} \neq 0$ on $\{x_t < \bar{x}\}$, and where the ergodic distribution places non-zero weight on the volatile region.

Proof of Proposition E.1. Let us conjecture an equilibrium of the form described in the proposition, i.e., the economy is stochastic for $x < \bar{x}$ and deterministic for $x \geq \bar{x}$.

By Itô's formula, we may derive the dynamics of π, which when combined with the linear Phillips curve (linear PC) yields the equation

$$\rho\pi(x) - \kappa x = \left[\bar{\imath} + \phi_x x + (\phi_\pi - 1)\pi(x) - \rho + \frac{1}{2}\sigma_x^2\right]\pi'(x) + \frac{1}{2}\sigma_x^2\pi''(x) \tag{E.5}$$

When $x \geq \bar{x}$, we have a deterministic equilibrium. In that case, equation (E.5) is an ODE for π, which has the solution $\bar{\pi}_0 - \bar{\pi}_1 x$ given in (E.4). Note that we pick the larger root for $\bar{\pi}_1$, but either choice would suffice here because both eigenvalues of the transition matrix $\mathcal{A}$ are strictly positive (hence unstable) for the parameter assumptions made here (see Proposition 6). Note also that $\bar{\pi}_1$ is real and positive by the parameter assumptions made.

For the stochastic region $x < \bar{x}$, we set $\pi(x)$ via (E.4), for some $\lambda \in (1, 2]$, and we pick σ_x^2 to ensure equation (E.5) holds exactly, i.e., set

$$\sigma_x(x)^2 = 2\frac{\rho\pi(x) - \kappa x - \left[\bar{\imath} + \phi_x x + (\phi_\pi - 1)\pi(x) - \rho\right]\pi'(x)}{\pi'(x) + \pi''(x)}, \quad \text{if } x < \bar{x}. \tag{E.6}$$

We must verify that σ_x^2 is non-negative and such that x_t follows a dynamically stable path.

Analogously to Proposition 2, we employ a change-of-variable and examine the dynamics of the level output gap $y = e^x$,

$$dy_t = \underbrace{y_t\left[\bar{\imath} - \rho + \phi_x \log(y_t) + (\phi_\pi - 1)\pi(\log(y_t)) + \sigma_x(\log(y_t))^2\right]}_{=\mu_y(y_t)} dt + \underbrace{y_t\sigma_x(\log(y_t))}_{=\sigma_y(y_t)} dZ_t \tag{E.7}$$

which is also a univariate diffusion process. The stability properties of y_t are determined by its boundary behavior as $y \to 0$ and $y \to e^{\bar{x}}$ (equivalently, $x \to -\infty$ and $x \to \bar{x}$).

First, we examine the boundary $y = 0$. Note that

$$\begin{aligned}\sigma_y(y)^2 &= y^2\sigma_x(\log(y))^2 \\ &= \frac{2y^2\left[\rho\hat{\pi}_0 y^\lambda + \rho\hat{\pi}_1 - \kappa\log(y)y^\lambda + \lambda\hat{\pi}_1\left(\bar{\imath} - \rho + \phi_x\log(y) + (\phi_\pi - 1)(\hat{\pi}_0 + \hat{\pi}_1 y^{-\lambda})\right)\right]}{\lambda(\lambda - 1)\hat{\pi}_1} \\ &\xrightarrow{y \to 0} \lim_{y \to 0} \frac{2(\phi_\pi - 1)\hat{\pi}_1 y^{2-\lambda}}{\lambda - 1}\end{aligned}$$

If $\lambda \in (1, 2)$ this limit is zero, and if $\lambda = 2$ the limit is a finite positive number, given

$\phi_\pi > 1$. On the other hand, the drift is

$$\mu_y(y) = y[\bar{\iota} - \rho + \phi_x \log(y) + (\phi_\pi - 1)\pi(\log(y))] + \frac{\sigma_y(y)^2}{y}$$
$$\xrightarrow{y\to 0} \lim_{y\to 0}(\phi_\pi - 1)\hat{\pi}_1 y^{1-\lambda}\Big(\frac{\lambda+1}{\lambda-1}\Big)$$

This limit is $+\infty$ for any $\lambda > 1$. Consequently, we have the following behavior. For $\lambda \in (1,2)$, the diffusion σ_y vanishes as $y \to 0$, while the drift μ_y explodes, which means the boundary $\{y = 0\}$ is inaccessible.

Second, examine the other boundary $y = e^{\bar{x}}$, where recall $\bar{x} < 0$. It is clear from the expression that $\sigma_y(e^{\bar{x}}-)$ is finite, whereas recall that $\sigma_y(e^{\bar{x}}) = 0$ by our construction. At the same time, the drift at that point is

$$\mu_y(e^{\bar{x}}) = e^{\bar{x}}\big[\bar{\iota} - \rho + \phi_x \bar{x} + (\phi_\pi - 1)(\bar{\pi}_0 - \bar{\pi}_1 \bar{x})\big]$$

This is negative if and only if $x < \bar{x}_{\max}$, which is guaranteed to hold by condition (E.1). Therefore, y_t can never exceed $e^{\bar{x}}$ if it starts below that point, i.e., if $y_0 \leq e^{\bar{x}}$.

Finally, we verify that σ_x^2 remains non-negative hence well-defined in the volatile region $x < \bar{x}$. Given the expression for σ_x^2, and the fact $\hat{\pi}_1 > 0$, $\phi_\pi > 1$, and $\lambda > 1$, this boils down the condition $G(y) > 0$, where

$$G(y) := \rho\hat{\pi}_0 y^\lambda + \rho\hat{\pi}_1 - \kappa\log(y)y^\lambda + \lambda\hat{\pi}_1\big(\bar{\iota} - \rho + \phi_x\log(y) + (\phi_\pi - 1)(\hat{\pi}_0 + \hat{\pi}_1 y^{-\lambda})\big)$$

Since $y < e^{\bar{x}} < 1$, we have $\log(y) < 0$ in this region, and so

$$G(y) > \hat{G}(y) := \rho\hat{\pi}_0 y^\lambda + \rho\hat{\pi}_1 + \lambda\hat{\pi}_1\big(\bar{\iota} - \rho + \phi_x\log(y) + (\phi_\pi - 1)(\hat{\pi}_0 + \hat{\pi}_1 y^{-\lambda})\big)$$

Thus, it suffices to show that $\hat{G}(y) > 0$ in this region. We will show, under the conditions provided, that $\hat{G}$ is monotonically decreasing and then that $\hat{G} > 0$ in this region.

Taking the derivative of $\hat{G}$, we have

$$\hat{G}'(y) = \lambda y^{-(\lambda+1)}\Big(\rho\hat{\pi}_0 y^{2\lambda} + \hat{\pi}_1\phi_x y^\lambda - \lambda\hat{\pi}_1^2(\phi_\pi - 1)\Big)$$

The equation $\hat{G}'(y) = 0$ is equivalent to a quadratic equation in y^λ and so has two roots. If $\hat{\pi}_0 < -\frac{\phi_x^2}{4\rho(\phi_\pi - 1)}$, then the two roots are complex, and so the condition $\hat{\pi}_1 > 0$ in (E.2) ensures that $\hat{G}'(y) < 0$ for all $y > 0$. If $\hat{\pi}_0 \geq -\frac{\phi_x^2}{4\rho(\phi_\pi - 1)}$, then the two roots are real and positive, and the condition $\hat{\pi}_1 > \frac{2\rho\hat{\pi}_0}{-\phi_x + \sqrt{\phi_x^2 + 4\rho\hat{\pi}_0(\phi_\pi - 1)}}$ in (E.2) ensures that both roots are larger than 1, i.e., $\hat{G}'(y) < 0$ for all $y \in (0,1)$. Either way, we have $\hat{G}'(y) < 0$ for all $y \in (0,1)$.

Given $\hat{G}$ is monotonically decreasing for $y \in (0,1)$, proving $\hat{G}(1) > 0$ ensures that $\hat{G} > 0$ in the entire volatile region $(0, e^{\bar{x}}) \subset (0,1)$. By condition (E.3), we have that $\hat{G}(1) > 0$, and so $\hat{G}(y) > 0$ for all $y \in (0,1)$. Thus, we have shown that $\sigma_x^2 > 0$ on $\{x \leq \bar{x}\}$ in our construction. This proves that we have a well-defined volatility function that keeps the dynamics stable in the region $\{x \leq \bar{x}\}$, and so a stationary distribution for x_t exists, meaning that Condition 1 is satisfied and the equilibrium is verified.

The only remaining question is whether $(\bar{x}, \hat{\pi}_0, \hat{\pi}_1)$ can be chosen to jointly satisfy conditions (E.1)-(E.3). This is straightforward to verify. For instance, by putting $\hat{\pi}_1 = e^{0.5\lambda\bar{x}}$ and $\hat{\pi}_0 > 0$ arbitrary, all three conditions are satisfied if $\bar{x}$ is made negative enough. (Alternatively, in the standard monetary case with $\bar{\iota} = \rho$, the conditions will hold with the combination of any choice for $\bar{x} < 0$, any choice for $\hat{\pi}_0 > 0$ and a sufficiently large choice for $\hat{\pi}_1$.) $\square$

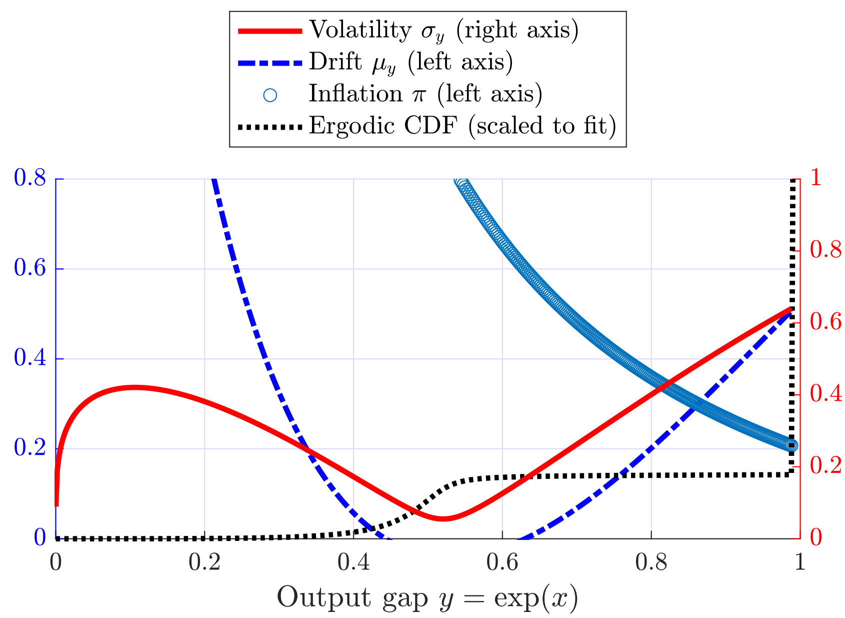

Figure E.1: Equilibrium with partially-flexible prices ($\kappa > 0$), a linear Taylor rule, a linearized Phillips curve, an inflation function in (E.4), and a volatility function in (E.6). The stationary CDF is computed via a discretized Kolmogorov Forward equation. The resulting stationary CDF features a mass point at $y = e^{\bar{x}}$. Parameters: $\rho = 0.02$, $\kappa = 0.2$, $\bar{\iota} = \rho$, $\phi_x = 0.8$, $\phi_\pi = 1.5$. The construction of the inflation function and volatile region uses $\bar{x} = -0.01$, $\lambda = 1.5$, $\hat{\pi}_0 = -0.2$, and $\hat{\pi}_1 = 0.4$.

F Zero Lower Bound

Let us address the fact that a zero lower bound (ZLB) constrains monetary policy. To simplify the exposition, we work exclusively in the rigid-price limit $\kappa \to 0$, and so inflation is zero ($\pi_t = 0$) and the nominal rate is equal to the real rate ($\iota_t = r_t$). To make matters interesting, we will assume that monetary policy aims to achieve the flexible-price allocation whenever possible, but they are subject to the ZLB $r_t \geq 0$.

In particular, monetary authorities set the nominal rate (hence the real rate) to implement $x_t = 0$ whenever possible, subject to the ZLB. This is the same idea behind the policy in Caballero and Simsek (2020), who consider a version of the New Keynesian model with risky capital. Under this policy rule, zero output gap prevails whenever the real rate is positive, and a negative output gap must arise at the ZLB (because recall raising the interest rate will lower output):

$$0 = \min[-x_t, r_t]. \tag{F.1}$$

In Lemma F.1 below, we show that within the class of equilibria we study, (F.1) is the outcome of optimal discretionary monetary policy (i.e., monetary policy without commitment to future policies). More deeply, the implementation of $x_t = 0$ "whenever possible" itself requires some kind of commitment to off-equilibrium threats, for instance to reduce interest rates if x_t ever fell below 0—this is the standard notion of "active" monetary policy that pervades the New Keynesian literature, but it becomes somewhat hidden by the outcome (F.1). In that sense, the rule (F.1) actually embeds some amount of commitment power.

Lemma F.1. *Optimal discretionary monetary policy—which maximizes (2) subject to $r_t \geq 0$, optimal household and firm decisions, and its own future decisions—implements (F.1).*

Proof of Lemma F.1. Since there is no upper bound on interest rates, the central bank can always threaten r_t high enough to ensure that $x_t \leq 0$. Since positive output gaps are undesirable, they will implement this. Then, we can restate the problem as: optimal discretionary monetary policy seeks to pick a r_t to maximize (2), subject to (IS), $x_t \leq 0$, the ZLB $r_t \geq 0$, and subject to its own future decisions.

We will discretize the problem to time intervals of length Δ and later take $\Delta \to 0$.

Noting that $C_t = e^{x_t} Y^*$, the time-t household utility is proportional to

$$\mathbb{E}_t\Big[\int_0^\infty \rho e^{-\rho s} x_{t+s} ds\Big] \approx \rho x_t \Delta + \mathbb{E}_t\Big[\int_\Delta^\infty \rho e^{-\rho s} x_{t+s} ds\Big]$$
$$\approx -\rho\Delta\mathbb{E}_t[x_{t+\Delta} - x_t] + \underbrace{\mathbb{E}_t\Big[\int_\Delta^\infty \rho e^{-\rho s} x_{t+s} ds\Big] + \rho\Delta\mathbb{E}_t[x_{t+\Delta}]}_{\text{taken as given by discretionary central bank}}.$$

The term with brackets underneath is taken as given by the time-t discretionary central bank, because it involves expectations of future variables that the future central bank can influence.

Thus, taking $\Delta \to 0$, the time-t central bank solves

$$\min_{r_t \geq 0} \mathbb{E}_t[dx_t]$$

subject to the constraints

$$r_t = \rho + \mu_{x,t} - \frac{1}{2}\sigma_{x,t}^2$$
$$x_t \leq 0 \quad \text{and if} \quad x_t = 0 \quad \text{then} \quad \mu_{x,t} = \sigma_{x,t} = 0.$$

Note that $\sigma_{x,t}$ is independent of policy when $x_t < 0$. There are two cases. If $x_t = 0$, then the constraints imply that $r_t = \rho$. If $x_t < 0$, we may substitute the dynamics of x_t (replacing μ_x from the first constraint) to re-write the problem as

$$\min_{r_t \geq 0} [r_t - \rho + \frac{1}{2}\sigma_{x,t}^2].$$

Since σ_x is taken as given, the optimal solution is $r_t = 0$. Thus, the discretionary central bank optimally sets

$$r_t = \rho \mathbf{1}_{\{x_t = 0\}}.$$

In other words, the complementary slackness condition $x_t r_t = 0$ holds, which together with $r_t \geq 0$ and $x_t \leq 0$ implies (F.1). $\square$

The entire model dynamics are characterized by the IS curve (IS) with volatility when $r_t = 0$ and $x_t < 0$ and deterministic dynamics otherwise, i.e.,

$$\mu_{x,t} = (-\rho + \frac{1}{2}\sigma_{x,t}^2)\mathbf{1}_{\{x_t < 0\}}. \tag{F.2}$$

The entire previous analysis from Section 3 goes through with $\phi_x = 0$ and $\bar{\iota} = 0$.

However, just to see a different construction, let $y = e^x$ and suppose

$$\sigma_x = \begin{cases} \nu(1-y), & \text{if } y < 1; \\ 0, & \text{if } y \geq 1. \end{cases} \tag{F.3}$$

(If we had set $\sigma_x = \nu/y$ when $y < 1$, then the argument would be identical to that in Section 3.) In this case, the dynamics of y_t are

$$dy_t = y_t\Big[-\rho + \nu^2(1-y_t)\Big]\mathbf{1}_{\{y_t<1\}}dt + y_t(1-y_t)\nu\mathbf{1}_{\{y_t<1\}}dZ_t. \tag{F.4}$$

This process never reaches $y = 0$, since it behaves asymptotically as a geometric Brownian motion as $y_t \to 0$. Thus, we have constructed a valid equilibrium with volatility at the ZLB.

If agents expect volatility to be sufficiently countercyclical, then the volatility is forever recurrent. To see this, suppose $\nu^2 > 2\rho$ so that $\log(y_t)$ has a positive drift as $y_t \to 0$. By standard arguments, y_t will not concentrate mass near $y = 0$ in the long run. On the other hand, the drift of $\log(y_t)$ is negative as $y_t \to 1$, and its volatility vanishes, so y_t will not ever reach $y = 1$ either. There will be a non-degenerate ergodic distribution of y_t, hence volatility $\sigma_{x,t}$. This economy is persistently demand-driven and stuck at the ZLB.

By adding coordinated jumps in σ_x, we believe we can make the equilibria even more realistic. Initially, volatility can be non-existent and the economy sitting at $x_t = 0$. All of a sudden, fear can rise sufficiently that x_t must jump to negative territory. Because of the ZLB, it is not possible for monetary policy to correct this fear-driven recession. The rise in volatility essentially forces r to the ZLB, similar to Caballero and Simsek (2020). Once $x_t < 0$, volatility can vary continuously, and sunspot shocks will be moving demand. Imagine at some later time T, demand reverts back to the flexible-price outcome $x_T = 0$. At some still later date, volatility can re-emerge. In this way, we can construct equilibria that alternate between efficiency and inefficient, self-fulfilling, volatile recessions.